Auctioning Heterogeneous Items: with Applications to Internet Advertisements and Privatizations

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Abstract

We analyze auctions with multiple similar, but heterogeneous items. Bidders’ preferences among the items are common, but bidders value the same item differently. Therefore, the model has both common value components and private value components. We show that the sequential sealed-bid first-price auction achieves both efficiency and revenue maximization simultaneously. This mechanism is used by a Japanese advertising company to sell internet keywords to sponsors and is very different from the mechanism used by the US search engines such as Google and Yahoo. The mechanism is also applicable to privatization problems.

Keywords: auctions, heterogeneous items, privatizations, internet advertisements.

1. Introduction

Sequential auctions are common mechanisms for selling multiple heterogeneous items. This paper shows that the first-price sealed-bid auction with the particular selling order achieves both efficiency and revenue maximization. This paper also characterizes the Bayesian perfect equilibrium strategies of the first-price sealed-bid auction and shows “the price decreasing anomaly” is obtained.

There are two important applications of this paper to practical problems. The first one is the governments’ privatization problems. For example, the Victorian State government started its privatization of power plants from 1993 and sold several state-owned power plants sequentially in a way that the largest plant was auctioned off last. This paper tells us that their order of sales is neither revenue maximizing nor efficient. Actually, the Victorian government should reverse the order of the sales and auction the largest plant first. Vickrey (1962), Weber (1983), and Milgrom (1997) are the closest to this paper. Vickrey (1962) analyzed auctions with common objects under the unit demand assumption. He characterized Nash equilibrium strategies for simultaneous auctions. Weber (1983) analyzed the same model as that of Vickrey and characterized the perfect Bayesian equilibrium strategies for sequential auctions. Milgrom (1997) is a special example of our model with only two items and a bidder’s valuations being multiplicative. He adopted Nash equilibrium as an equilibrium concept and showed that a sequential sealed-bid first-price auction implements the optimal allocation mechanism. This paper differs from his work in important ways. Firstly, we adopt the Bayesian perfect equilibrium as an equilibrium concept instead of the Nash equilibrium. Secondly, our model analyzes \( m \geq 2 \) items case and bidders utility.
functions are much more general than a multiplicative function. Lastly, we investigate the price patterns of sequential auctions, which Milgrom (1997) did not pay any attention at all. It is worth noting that Baba (1999) characterized the mechanism which achieves efficiency, expected revenue maximization, and equal treatment of the winners in the sense that each winner pays the same per-unit price regardless of which item he wins.

The second application is the internet keyword auctions. Edelman et al. (2007), Milgrom (2010), and Varian (2007) focus on the generalized second-price auctions and show that it can implement the expected revenue maximizing auction and achieves an efficient allocation, but we show that the first price auction also serves the same purpose. Furthermore, the first price auction analyzed in this paper is actually used in D2 Com munications in Japan to sell internet keywords.

The rest of the paper is organized as follows: Section 2 explains the model. Section 3 analyzes the sequential sealed-bid first-price auction. Section 4 characterizes the optimal auction mechanism and addresses how to implement it. Section 5 summarizes the result, the advancement of business, shows several directions to extend the basic model, and discuss future research prospects. All the proofs are collected in the appendices.

2. The Model

There are \( n+1 \geq m \geq 1 \) potential buyers for \( m \) objects. \( N \) is the set of potential buyers and \( M \) is the set of items. Each buyer \( i \) draws his private signal \( x_i \) independently from some common probability distribution \( F(\bullet) \) whose support is \( [0, \ x] \) which has a continuous density function \( f(\bullet) \). There are \( m \) common public signals, \( t_1 \geq t_2 \geq \ldots \geq t_m \). When buyer \( i \) with private signal \( x_i \) obtains the \( k \)th item at price \( p_k \), he receives net utility of \( U(t_k, x_i) - p_k \) where \( U(t_k, x_i) \) is a supermodular. We also assume that each potential buyer \( i \) demands at most one object, or equivalently, that the seller restricts each buyer to obtain at most one object. The structure of a game is common knowledge among the players except each bidder only knows his realized private signal but not others.

First, we analyze sequential sealed-bid first-price auctions with price announcement. There, the seller auctions one object at each period following a predetermined order. At the beginning of each period, the seller announces the price paid in the previous period. This allows all losing bidders to form the same belief on their opponents’ private signals at the beginning of each period. Further, the bidder who submits the highest bid obtains the object and pays the amount of his own bid. It is possible to consider other information releasing procedures such as no announcement at all about the price of the previous period’s auction. In this case, losing bidders only know that the fact that they “lost.” However, it is worth noting that we can show that bidders’ strategies remain the same on the equilibrium path as the case with price announcement.

We adopt the Bayesian perfect equilibrium as an equilibrium concept and analyze symmetric equilibrium strategies. A strategy \( b \) or a bidder, \( b_k(x; \bullet) (\forall k \in M) \), is a combination of functions whose domain is \( R \times R^{k-1} \) and whose range is \( R \). The first argument indicates his
realized private signal and the second represents any information available up to period \( k \) such as the realized prices in the previous auctions. In other words, \( b_k(x; \bullet) \) indicates how much a bidder is willing to bid in the period \( k \) auction when he loses all the auctions prior to period \( k \). Later, we need to consider each bidder's belief on their opponents' private signals at the beginning of each period to characterize equilibrium bidding strategies. For this purpose, we need additional notation. \( X_i \) is a random variable of bidder \( i \)'s private signal \( (\forall i \in N) \) and \( x_i \) is its realization. Further, \( Y_r \) \((r = 1, 2, ..., n)\) is the \( r \)th higher order statistics of \( X_1, X_2, ..., X_{n+1} \) and \( y_r \) is a realization of \( Y_r \).

3. Sequential Sealed-Bid First-Price Auctions

First, we analyze the model with price announcement, that is, the seller announces the price of the previous auction at the beginning of each period before bidders submit their bids. This allows all the remaining bidders to have the same belief on the opponents’ private signals in each period. The following proposition characterizes the symmetric Bayesian perfect equilibrium strategies in this case.

**Proposition 1 (with price announcement):**
Monotonically increasing equilibrium bidding functions \( b_k(x; y_1, y_2, ..., y_{k-1}) = b_k(x) \) \((\forall k \in M)\) exist if the objects are sold by the predetermined order of \( t_1 \geq t_2 \geq ... \geq t_m \) and characterized as follows:

If \( y_{k-1} \geq x \),

\[
\begin{align*}
b_k(x) &= \int_0^{(n-k+1)} \left( \frac{F(s)}{F(x)} \right)^{n-k+1} \left( \frac{f(s)}{F(x)} \right) \left( (U(t_k, x) - U(t_{k-1}, x))s + b_{k-1}(s) \right) ds \quad \text{for } 1 \leq k \leq m - 1 \\
b_m(x) &= \int_0^{U(t_m, s)(n-m+1)} \left( \frac{F(s)}{F(x)} \right)^{n-m+1} \left( \frac{f(s)}{F(x)} \right) ds
\end{align*}
\]

Otherwise, \( b_k(x) = b_k(b_k^{-1}(p_{k-1})) \) \( \text{for } 1 \leq k \leq m \).

**Proof:**
See Appendix.

We can immediately obtain the following corollary from proposition 1.

**Corollary 1 (without price announcement):**
Suppose that the seller does not announce the price of the auctions at all. Then, there exists equilibrium bidding functions that are the same as those in Proposition 1 on the equilibrium path.

Next, we define unadjusted and adjusted prices to examine the price path.

**Definition (unadjusted price):**
The unadjusted price in period \( k \) \((\forall k \in M), p_k \), is the winning bid of the period \( k \) auction for the sequential sealed-bid first-price auction and the highest loser’s bid \( f \) or the sequential sealed-bid second-price auction.
**Definition (adjusted price):**
The adjusted price at period $k$ ($\forall k \in M$) is defined as the unadjusted price divided by $t_k$ (i.e., $\frac{P_k}{t_k}$) when $U(t_k, x) = t_k x$.

The following proposition and corollary examine the pattern of the price paths.

**Proposition 2 (declining adjusted prices):**
In the sequential sealed-bid first-price auction with predetermined selling order of $t_1 \geq t_2 \geq \ldots \geq t_m$, the unconditional expected adjusted prices decline on the equilibrium path.

**Proof:**
See Appendix.

We can immediately obtain the following corollary from proposition 2.

**Corollary 2 (declining unadjusted prices):**
In the sequential sealed-bid first-price auction with predetermined selling order of $t_1 \geq t_2 \geq \ldots \geq t_m$, the unconditional expected adjusted prices decline on the equilibrium path.

4. The Expected Revenue Maximizing Auction Mechanism

This section characterizes the expected revenue maximizing (optimal) auction mechanism in our setting and shows that both the sequential sealed-bid first-price and second-price auctions implement the optimal auction mechanism. First, we characterize the optimal auction mechanism. To simplify the analysis, we impose the following regularity condition.

**Regularity condition:**
The regularity condition is satisfied if $U(t_k, x_i) - \left( \frac{\partial U(t_k, x_i)}{\partial x_i} \right) \left( 1 - \frac{F(x_i)}{f_i(x_i)} \right)$ increases w.r.t. $x_i$, $\forall i \in N$ and $k \in M$.

**Proposition 3 (the optimal mechanism):**
Assume that the regularity condition holds and bidders are symmetric ex-ante. Then, the optimal auction mechanism is characterized by the following three conditions:

1. The payment of player $i$ ($\forall i \in N$) who reports his private signal as $x_i$ is
   $$c(x_i) = \mathbb{E} \left[ \sum_{k=1}^m U(t_k, x_i) p_n^*(\bar{X}) - \int_0^{x_i} \frac{\partial U(t_k, s)}{\partial t_k} p_n^*(s, \bar{X}_-^i) ds \right] | X_i = x_i,$$

   where,
   $$\bar{X} = (X_1, X_2, \ldots, X_{n+1}), \quad \bar{x} = (x_1, x_2, \ldots, x_{n+1}), \quad \bar{X}_- = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{n+1})$$

2. $p_n^*(x) = \begin{cases} 1 & \text{if } x \text{ is the kth highest among } \bar{x} \\ 0 & \text{otherwise} \end{cases}$
(3) The required minimum reported private signal is
\[ x'_k \quad (\forall k \in M) \] s.t.
\[ U(t_k, x'_k) - \left( \frac{\partial U(t_k, x)}{\partial x} \right)_{x=x'_k} \left( 1 - F(x'_k) \right) f(x'_k) = 0. \]

**Proof:**
We can obtain the result by slight modification of the argument in Myerson (1981).

**Proposition 4 (implementation):**
The sequential sealed-bid first-price auctions with or without price announcement with the minimum required reported private signal \( x'_k \quad (\forall k \in M) \) implement the optimal selling mechanism.

**Proof:**
Obvious from propositions 1 and 3.

**5. Discussion**

This paper demonstrates that the sequential first-price sealed-bid auction with the particular order of sales is the best procedure to privatize multiple heterogeneous items characterized by common component and private component. To this purpose, we characterize the Bayesian perfect equilibrium bidding functions of the sequential sealed-bid first-price auctions and shows that they can achieve an efficient allocation and maximizes the expected revenue when more valuable objects are auctioned earlier. In this sense, this paper serves as a guidebook to many governments facing various privatization problems: which auction procedures should be adopted, which items should be auctioned off earlier, how to set reserve prices, and how much money they can raise.

Secondly, our mechanism is also applicable to the hot issue of internet keyword advertisement auctions analyzed by Edelman et al. (2007), Milgrom (2010), and Varian (2007). Although all three papers focus on the generalized second price auctions, this paper shows that the first price auction also can implement the expected revenue maximizing auction and achieve an efficient allocation. Furthermore, the first price auction analyzed in this paper is actually used in D2 Com munications in Japan to sell internet keywords. Lastly, in addition to achieving an efficient allocation and maximizing expected revenue, the mechanism analyzed by Baba (1999) treats the winners equally in the sense that each winner pays the same per-click price regardless of the slot he wins.

Thirdly, decreasing prices anomaly is observed in the sequential sealed-bid first-price auctions. Although literature assumed some kind of homogeneity across items, either in deterministic or stochastic sense to explain decreasing price anomaly, we believe that the seller’s choosing the sales order of heterogeneous items is another source of observing decreasing price anomaly in the practical world. Note that “prices decreasing anomaly” implies unequal treatment of the winners and it causes serious problem when the government designs a selling procedure to privatize state-owned properties. For example, in the Japanese land auctions, it means that different winners pay different per square meter price for the adjacent pieces of land and it is sometimes not approval.
There are a few important directions for future research. First, though we assume that bidders’ private signals are i.i.d., this should be only the first step to analyze more practical cases such as affiliated valuations. Next, there are exam ples in the practical world where each bidder demands more than one objects. Thirdly, we also need to consider the interdependency across different objects to characterize bidders’ strategies. The FCC radio spectrum license auctions are one of the most important examples among others which have all three characteristics. This will complicate the analysis, but be worth challenging.

6. References


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Appendix

Proof of Proposition 1:
We use mathematical induction to prove Proposition 1. There are four steps in the argument. The first two steps characterize bidding functions on the equilibrium path. The third step examines the off-the-equilibrium-path strategies. The fourth step shows that the strategies in Proposition 1 attain global maximum.
Step 1 (period $m$).
Without loss of generality, we can focus on the problem for bidder 1. Assume that bidder 1 loses all the $(m-1)$ auctions held prior to period $m$ and consider bidder 1’s problem for the last period (period $m$) auction. Since we assume that the seller announces the price of the previous period auction at the beginning of the next before bidders submit their bids, all the remaining bidders have the same belief on the opponents’ private signals at the beginning of the last period’s auction. Namely, all the bidders know the value of the winning bids up to period $m-1$, which we denote by $b_1^w, b_2^w, ..., b_{m-1}^w$.

On the equilibrium path, this means that the remaining bidders actually know the value of $y_1, y_2, ..., y_{m-1}$ from $b_1^w, b_2^w, ..., b_{m-1}^w$ by inverting bidding functions, since we assume that there exists monotonically increasing equilibrium bidding strategy w.r.t. private signal. This means that all the remaining bidders know that all the remaining opponents’ private signals are i.i.d. from the probability distribution function $\frac{F(.)}{F(y_{m-1})}$. So, the bidder 1’s problem in period $m$ auction is the same as one in the situation where there are $(n+1)-(m-1)$ potential buyers whose private signals are i.i.d. of the truncated probability distribution function $\frac{F(.)}{F(y_{m-1})}$.

Therefore, we can write bidder 1’s problem as follows:

$$\max_{x_m} \left( \frac{F(x_m)}{F(y_{m-1})} \right)^{n-m+1} (U(t_m, x) - b_m(x_m)) \quad \ldots (1)$$

The F.O.C. w.r.t. $x_m$ of (1) evaluating at $x_m = x$ yields

$$(n-m+1) \left( \frac{F(x)}{F(y_{m-1})} \right)^{n-m} \left( \frac{f(x)}{F(y_{m-1})} \right) (U(t_m, x) - b_m(x)) - \left( \frac{F(x)}{F(y_{m-1})} \right)^{n-m} b_m'(x) = 0 \quad \ldots (2)$$

Solving (2) with boundary condition of $b_m(0) = 0$ leads to

$$b_m(x) = \int_0^x U(t_m, s)(n-m+1) \left( \frac{F(s)}{F(x)} \right)^{n-m} \left( \frac{f(s)}{F(x)} \right) ds \quad \ldots (3)$$

Step 2 (period $k=1, 2, ..., m-1$).
Suppose that there exist monotonically increasing bidding functions from period $k+1$ to period $m$, $b_{k+1}(x), b_{k+2}(x), ..., b_m(x)$, which do not depend on the realization of the price path. Then, we would like to show that there exists a monotonically increasing bidding function in period $k$, $b_k(x)$, which does not depend on the realization of the price path, either.

Bidder 1’s problem is expressed as follows:

$$\max_{x_k} U_k(x_k, x; \ Y_1 = y_1, Y_2 = y_2, ..., Y_{k-1} = y_{k-1})$$

where, $U_k(x_k, x; \ Y_1 = y_1, Y_2 = y_2, ..., Y_{k-1} = y_{k-1})$ is the expected utility of bidder 1 when his true private signal is $x$ and his reported private signal is $x_k$, given that $Y_1 = y_1, Y_2 = y_2, ..., Y_{k-1} = y_{k-1}$. The first order condition w.r.t. $x_k$ for the case of $x_k > x$ and for the case of $x_k < x$ evaluated at $x_k = x$ coincide and is expressed as follows.

$$b_k'(x) + (n-k+1) \frac{f(x)}{F(x)} b_k(x) = (n-k+1) \left( \frac{f(x)}{F(x)} \right) (U(t_k, x) - U(t_{k+1}, x) + b_{k+1}(x)) \quad \ldots (4)$$
Solving (4) for \( b_k(x) \) with the boundary condition \( b_k(0) = 0 \) yields
\[
b_k(x) = \int_0^x (n - k + 1) \left( \frac{F(s)}{F(x)} \right)^{n-k} \left( \frac{f(s)}{F(x)} \right) (U(t_k, s) - U(t_{k+1}, s) + b_{k+1}(s)) ds \tag{5}
\]
Together with (3), the bidding functions \( b_k(x) \) (\( \forall k \in M \)) are recursively determined as follows:
\[
b_m(x) = \int_0^x U(t_m, x) \left( \frac{F(s)}{F(x)} \right)^{n-m+1} \left( \frac{f(s)}{F(x)} \right) ds
\]
or
\[
b_k(x) = \int_0^x (n - k + 1) \left( \frac{F(s)}{F(x)} \right)^{n-k+1} \left( \frac{f(s)}{F(s)} \right) (U(t_k, s) - U(t_{k+1}, s) + b_{k+1}(s)) ds , \text{ for } 1 \leq k \leq m-1
\]

**Step 3 (off the equilibrium path).**
If the sequence of the realized values of \( y_1, y_2, ..., y_{k-1} \) obtained from inverting \( b_1^w, b_2^w, ..., b_{k-1}^w \) is not decreasing, let bidders re-order \( y_1, y_2, ..., y_{k-1} \) to yield a decreasing sequence of \( z_1, z_2, ..., z_{k-1} \) and apply the argument in step 1 and 2 to the sequence of \( z_1, z_2, ..., z_{k-1} \). Since we characterize the Bayesian perfect equilibrium, this procedure works.

**Step 4 (Global maximum).**
The global optimality follows from the usual argument and it is straightforward. Q.E.D.

The following two lemmata are useful to prove proposition 2.

**Lemma 1 (Weber 1983):**
Suppose that there are \( n+1 \) bidders and that \( m \) homogeneous objects are up for sequential sealed-bid first-price or second-price auction. Then unconditional expected value of the prices remain the same for all the periods for both types of auction.

**Lemma 1’:**
Suppose \( \exists k \in \{1, 2, ..., m-1\}, t_k = t_{k+1} \geq t_{k+2} \geq ... \geq t_m \). Then, the unconditional expected values of both the adjusted and the unadjusted prices are the same for period \( k \) and period \( (k+1) \) for both sequential sealed-bid first-price and second-price auctions.

**Proof of lemma 1’:**
We can obtain the results from plugging \( t_k = t_{k+1} \) in the proof of proposition 1. Q.E.D.

**Proof of Proposition 2:**
We use Lemma 1 later. First, recall that proposition 1 tells us the equilibrium bidding functions in our case are:
\[
b_k(x) = \int_0^x (n - k + 1) \left( \frac{F(s)}{F(x)} \right)^{n-k+1} \left( \frac{f(s)}{F(s)} \right) (t_k - t_{k+1}) ds + b_{k+1}(s) ds \tag{5}
\]
Similarly,
\[ b_{k+1}(x) = \int_0^x (n-k+1) \left( \frac{F(s)}{F(x)} \right)^{n-k} \left( \frac{f(s)}{F(s)} \right) \left( t_{k+1} - t_{k+2} \right) ds + b_{k+2}(s) ds \]

for \( k = 1, 2, \ldots, m - 2 \) \( \ldots(6) \)

Differentiate (6) w.r.t. \( t_{k+1} \),

\[ \frac{\partial b_{k+1}(x)}{\partial t_{k+1}} = \int_0^x (n-k) \left( \frac{F(s)}{F(x)} \right)^{n-k} \left( \frac{f(s)}{F(s)} \right) s \ ds \ \ldots(7) \]

Using (7),

\[ \frac{\partial b_{k+1}(x)}{\partial t_{k+1}} = \frac{1}{t_{k+1}^2} \left( \frac{\partial b_{k+1}(x)}{\partial t_{k+1}} \right) t_{k+1} - b_{k+1}(x) \]

\[ \frac{\partial b_{k+1}(x)}{\partial t_{k+1}} = \frac{1}{t_{k+1}^2} \left( \frac{\partial b_{k+1}(x)}{\partial t_{k+1}} \right) t_{k+1} - b_{k+1}(x) \]

\[ = \frac{1}{t_{k+1}^2} \left\{ \int_0^x (n-k) \left( \frac{F(s)}{F(x)} \right)^{n-k} \left( \frac{f(s)}{F(s)} \right) \left( t_{k+1} - t_{k+2} \right) s + t_{k+2} s - b_{k+2}(s) ds \right\} > 0 \ \ldots(8) \]

On the other hand,

\[ \frac{\partial b_{k+1}(x)}{\partial t_{k+1}} = \frac{1}{t_{k+1}} \left( \frac{\partial b_{k+1}(x)}{\partial t_{k+1}} \right) = \int_0^x (n-k+1) \left( \frac{F(s)}{F(x)} \right)^{n-k+1} \left( \frac{f(s)}{F(s)} \right) \left( -s + \frac{\partial b_{k+1}(s)}{\partial t_{k+1}} \right) ds \ \ldots(9) \]

Further,

\[ \left( -s + \frac{\partial b_{k+1}(s)}{\partial t_{k+1}} \right) = -s + \int_0^x (n-k) \left( \frac{F(t)}{F(x)} \right)^{n-k} \left( \frac{f(t)}{F(t)} \right) t \ dt \]

\[ = \int_0^x (n-k) F(t)^{n-k+1} f(t) t \ dt \]

\[ = \int_0^x (n-k) \left\{ F(t)^{n-k} \int_0^t F(t)^{n-k} dt \right\} \]

\[ = -s + s - \int_0^t F(t)^{n-k} dt < 0 \ \ldots(10) \]

From (8) and (10),

\[ \frac{\partial b_{k+1}(x)}{\partial t_{k+1}} < 0 \ \ldots(11) \]

From (8), (11) and Lemma 1', we can obtain the desired result as follows. First, Lemma 1' tells us that the difference between unconditional expected adjusted prices of the period \( k \) and \( (k+1) \) auctions equals to 0 when \( t_k = t_{k+1} \), or equivalently, \( \frac{t_{k+1}}{t_k} = 1 \). Further, (8) and (11) imply that the difference between unconditional expected adjusted prices of the period \( k \) and \( (k+1) \) auctions is a decreasing function w.r.t. \( \frac{t_{k+1}}{t_k} \). Therefore, we know that the difference between unconditional expected adjusted prices of the period \( k \) and \( (k+1) \) auctions are strictly positive as long as \( t_k > t_{k+1} \) and this is the desired conclusion. Q.E.D.