Applying Financial Rigour to Fuzzy Real Options

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Fuzzy numbers have recently been introduced in the real options literature as a simple alternative to option valuation. However, so far the assumption that the project value is a fuzzy number has not been put on solid theoretical ground. Here, we consider two methods to value a Real Options project whose future expected cash flows are based on managerial triangular estimates. Both methods rely on correlating a geometric Brownian motion (GBM) process to a traded market security or index. By utilizing the minimal entropy martingale measure, we value the real option in a theoretically sound manner.

1 Introduction

Real option analysis (ROA) has been recognized as a superior method to quantify the value of real-world investment opportunities where managerial flexibility can influence their worth, compared to standard net present value (NPV) discounted cash flow (DCF) analysis. ROA builds on the seminal work of Black and Scholes (1973) on financial option valuation. Myers (1977) recognized the analogy between financial options and project decisions; both are exercised after uncertainties are resolved. Early academic techniques took this comparison literally; the Black-Scholes equation was applied to value put and call options on tangible assets. Since this early connection, ROA has been popularized by business publications and valuation texts, and in the last decade has transitioned from academic circles to heightened industry attention (see for example Borison (2005)). Real option analysis is a relatively new paradigm in corporate and project appraisal, a departure from traditional static appraisal, and it bridges concepts of valuation and strategic planning.

Despite the theoretical appeal of real option analysis, the Block (2007) survey of the Fortune 1000 largest companies found only 11.9% of respondents use the method. Arguably management realizes the benefit of ROA, but the adoption within industry is limited due to the complexity of analytical solutions, the restrictive assumptions required, and overall lack of intuition. To strengthen appeal and gain acceptance from academics and practitioners, Copeland and Antikarov (2001) introduced the Market Asset Disclaimer (MAD) method. While the method is somewhat straightforward to use, requiring only a basic understanding of the binomial option valuation method, the theoretical basis for the model is still debated (Borison (2005)).

Another practical approach for ROA, the Datar-Mathews method (Datar and Matthews (2004), Matthews and Datar (2007)), was introduced to help bridge the gap between theoretical models and practical implementation. The method relies on utilizing Monte Carlo simulation to determine the real option value based on simple probability distribution cash-flow projections. The major
contribution of the Datar-Mathews method is that it relies on cash flow scenarios that managers are comfortable in projecting; namely, it relies on pessimistic, likely and optimistic forecasts. Collan, Fullr, and Mezei (2009) extend the Datar-Mathews method in a practical fuzzy numbers context.

Two methods are discussed here that utilize the practicality of relying on managerial cash flow forecasts based on pessimistic, most likely and optimistic values, but add financial rigour consistent with financial theory. Two key aspects that are addressed include: 1) the probability distribution of the cash flows are linked to each other, and 2) the models allow for the estimated cash flows to be appropriately accounted for with respect to market versus idiosyncratic risk. The first method assumes a GBM process for the cash flows and links managerial expectations by assuming different growth and volatility values between time periods. The second method relies on deriving the value of each projected cash flow stream as a function of an underlying observable, but not necessarily traded, stochastic process. As an example, the stochastic process could be assumed to be the market size for a given product line. Based on historical data, market growth and correlation to a traded index of the market indicator can easily be determined. In both cases, since the cash flows are linked to a stochastic process, we can be assured that the analytics properly account for the correlation between the cash flow at say, time, \( t_i \) and some other future time, \( t_{i+n} \). Furthermore, correlation estimates between the underlying stochastic processes and a traded index allow for proper accounting of systematic versus idiosyncratic risk, consistent with the Capital Asset Pricing Model.

The following section will briefly present the density and distribution functions for triangular cash flow payoffs. Next, the first method of valuation, based on the assumption that the cash flows follow a GBM process will be introduced. A sketch of the second method will be presented afterwards, the details of which will be presented in a separate publication. Next, results of applying both methods to a hypothetical example with special emphasis given to the valuations for varying systematic versus idiosyncratic risk assumptions, will be given. The final section will provide concluding remarks.

2 Fuzzy Numbers for Cash Flow Estimates

A fuzzy number \( A \), properly normalized, can be reinterpreted as a random variable with a triangular density function

\[
f_A(y) = \begin{cases} 
    \frac{y - y_-}{y_0 - y_-}, & y_- < y \leq y_0, \\
    \frac{y_0 - y}{y_+ - y_0}, & y_0 < y \leq y_+, \\
    0, & \text{otherwise.}
\end{cases}
\]

(1)
Here, the normalizing constant \( a = 2/(y_+ - y_-) \). The corresponding distribution function is

\[
F_A(y) = \begin{cases} 
0, & y \leq y_- , \\
\frac{a \ (y - y_-)^2}{2 \ y_0 - y_-} , & y_- < y \leq y_0 , \\
1 - \frac{a \ (y_+ - y)^2}{2 \ y_+ - y_0} , & y_0 < y \leq y_+ , \\
1 , & y > y_+ .
\end{cases}
\]  

(2)

Figure 1 shows a typical configuration. Note that \( y_- , y_0 \) and \( y_+ \) carry the interpretation of the project value in a pessimistic or poor scenario, a likely or moderate scenario and an optimistic or good scenario. Consequently, the fuzzy nature of the project value can be viewed as placing the simplest distribution on these scenarios.

![Triangular Probability Density Function](image)

Figure 1: A fuzzy number can be reinterpreted as a random variable with triangular pdf.

A project manager can also provide a time component to the fuzzy nature of the project value by providing estimates of poor, moderate and good at a sequence of dates (e.g. annually). This is depicted in Figure 2. This sequence of estimates can be interpolated at finer time scales if investment decisions are to be made at these finer time scales, for example, if annual estimates are provided, but the decision to invest in the project can be made on a monthly basis.

3 Method 1: Valuation Through Fitting Cash Flows

Our main goals is to provide a financially consistent dynamic model which leads to a project valuation based on managerial estimates of cash flows consisting of pessimistic, likely and optimistic values, with the ability to account for systematic and idiosyncratic risk. In the model formulation, we assume that a significant investment at time, \( t \), of value, \( K \), will be required to receive the future cash flows at times \( t_1 , t_2 , ... , t_n \), where \( t \leq t_1 \). We assume that the cash flow process follows GBM of the following form

\[
df_t = \mu_t f_t dt + \sigma_t f_t dW_t ,
\]

(3)

where the process has discontinuous growth, \( \mu_t \), and volatility, \( \sigma_t \), parameters corresponding to time intervals \( t_{i-1} \) to \( t_i \). It should be emphasized, however, that we make the assumption that
the cash flows are received continuously, and thus, $f_t$ may be thought of as a cash flow rate. The parameters $\mu_i$ and $\sigma_i$ can be chosen to fit the pessimistic, likely and optimistic managerial estimates, by recognizing that

$$f_t = f_0 e^{\left(\frac{\mu_1 - \sigma_1^2}{2}\right)t_1 + \left(\frac{\mu_2 - \sigma_2^2}{2}\right)(t_2-t_1) + \ldots + \left(\frac{\mu_i - \sigma_i^2}{2}\right)(t_i-t_{i-1}) + \sqrt{\sigma_1^2 t_1 + \sigma_2^2 (t_2-t_1) + \ldots + \sigma_i^2 (t_i-t_{i-1})}Z}$$

with $Z \sim N(0,1)$. Clearly, $f_t$ is not a traded entity, but, we assume an index exists following another GBM process such that

$$dI_t = \nu I_t dt + \eta I_t \left(\rho dW_t + \sqrt{1-\rho^2} d\widehat{W}_t\right)$$

where the notation $W_t$ is used to represent a second, independent Wiener process. Under the risk-neutral measure, the index process is written as

$$dI_t = r I_t dt + \eta I_t \left(\rho d\widehat{W}_t + \sqrt{1-\rho^2} d\widehat{W}_t\right)$$

where $r$ is the risk-free rate. The cash flow process can then be written as

$$df_t = \bar{r}_i f_t dt + \sigma_i f_t d\widehat{W}_t$$

where $\bar{r}_i = \mu_i - \frac{\partial \sigma_i}{\partial \eta} (\nu - r)$ in the case where the correlation of the cash flows to the traded index changes in every period, or $\bar{r}_i = \mu_i - \frac{\sigma_i}{\eta} (\nu - r)$ in the case where $\rho$ remains constant.

Now, under the risk-neutral measure, the cash flow process becomes

$$\hat{f}_t = f_0 e^{\left(\bar{r}_1 - \frac{\sigma_1^2}{2}\right)t_1 + \left(\bar{r}_2 - \frac{\sigma_2^2}{2}\right)(t_2-t_1) + \ldots + \left(\bar{r}_i - \frac{\sigma_i^2}{2}\right)(t_i-t_{i-1}) + \sqrt{\sigma_1^2 t_1 + \sigma_2^2 (t_2-t_1) + \ldots + \sigma_i^2 (t_i-t_{i-1})}Z}.$$

The value of the cash flow process at time, $t$, is given by

$$V_t = \mathbb{E} \left[ \int_t^T e^{-r(s-t)} \hat{f}_s ds \mid \mathcal{F}_t \right]$$

$$= \int_t^T e^{-r(s-t)} \mathbb{E} \left[ \hat{f}_s \mid \mathcal{F}_t \right] ds.$$
But \( \mathbb{E} \left[ \hat{f}_s \mid \mathcal{F}_t \right] = \hat{f}_t e^{\hat{r}_1 (t_1 - t) + \hat{r}_2 (t_2 - t_1) + ... + \hat{r}_n (t_n - t_{n-1})} \), so that

\[
V_t = \int_t^{t_1} \hat{f}_t e^{\hat{r}_1 (s-t) - r(s-t)} ds + \int_{t_1}^{t_2} \hat{f}_t e^{\hat{r}_1 (t_1 - t) + \hat{r}_2 (s-t_1) - r(s-t)} ds + \ldots + \int_{t_{n-1}}^{t_n} \hat{f}_t e^{\hat{r}_1 (t_1 - t) + \hat{r}_2 (t_2 - t_1) + \ldots + \hat{r}_n (s-t_{n-1}) - r(s-t)} ds.
\]

(10)

Furthermore, \( \hat{f}_t = f_0 e^{\left( \hat{r}_1 - \frac{\sigma_t^2}{2} \right) t + \sigma_t \sqrt{t} Z} \), therefore equation (10) gives

\[
V_t = f_0 e^{\left( \frac{-\sigma_t^2}{2} \right) t + \sigma_t \sqrt{t} Z} \left( e^{(\hat{r}_1 - r)t_1} + e^{(\hat{r}_2 - r)t_2} - e^{(\hat{r}_2 - r)t_1} + \ldots + \frac{e^{\hat{r}_1 t_1 + \hat{r}_2 t_2 + \ldots + \hat{r}_n t_n} - e^{(\hat{r}_n - r)t_n}}{\hat{r}_n - r} \right)
\]

(11)

or

\[
V_t = V_0 e^{\left( \frac{-\sigma_t^2}{2} \right) t + \sigma_t \sqrt{t} Z}
\]

(12)

where

\[
V_0 = f_0 e^{\left( \frac{(\hat{r}_1 - r)t_1}{{\hat{r}_1} - r} + e^{(\hat{r}_1 - r)t_2} - e^{(\hat{r}_2 - r)t_1} + \ldots + \frac{e^{\hat{r}_1 t_1 + \hat{r}_2 t_2 + \ldots + \hat{r}_n t_n} - e^{(\hat{r}_n - r)t_n}}{\hat{r}_n - r} \right)}
\]

(13)

The value of having the option to make an investment, \( K \), at time, \( t \), to receive the future cash flows, as discussed above can now be written as

\[
O_0 = e^{-rt} \mathbb{E} \left[ \max (V_t - K, 0) \right].
\]

(14)

Given the form of \( V_t \) (equation (12)), we see that the value of the option simplifies to the standard Black-Scholes call option equation,

\[
O_0 = V_0 \Phi(d_1) - Ke^{-rt} \Phi(d_2)
\]

(15)

where

\[
d_1 = \frac{\ln \left( \frac{V_0}{K} \right) + \left( r + \frac{\sigma_t^2}{2} \right) t}{\sigma_t \sqrt{t}}, \quad d_2 = d_1 - \sigma_t \sqrt{t}.
\]

(16)

### 4 Sketch of Method 2: Valuation Through Matching Cash Flows

In the application of this method, we assume that there exists a market sector indicator, \( S_t \), that follows a GBM of the form,

\[
dS_t = \mu S_t dt + \sigma S_t dW_t,
\]

(17)
where, as before, $W_t$ is a standard Brownian motion under the real-world measure. The market sector indicator, $S_t$, which, for example, can be thought of as industry sector market size, is not itself tradable, but we assume it is correlated with a market index, $I_t$, similar to that of equation (6). It can be shown that a function $\varphi(S_t; S_0)$ exists so that each $i$-th triangular payoff, $f_{Ai}(y)$, can be matched with $\varphi(S_t; S_0)$, and its value, $V_{it}$, can be derived. Figure 3 shows that under the risk-neutral measure, the value of the option to invest in the project can then be calculated similarly to equation (14),

$$O_0 = e^{-rtE} \max \left( V_t - K, 0 \right).$$

where, again, similarly to the formulation above,

$$V_t = \sum_{i=1}^{n} e^{-r(t_i-t)}E \left[ V_{ti} | F_t \right],$$

and the total project value becomes

$$V_{proj} = e^{-rtK} \int_{-\infty}^{\infty} \max \left( \sum_{i=1}^{n} \left( \varphi_i \left( S_{ti}; S_0 e^{(\bar{r}-\frac{1}{2}\sigma^2)t_K + \sigma \sqrt{t_K} x} \right) e^{-r(t_i-t_K)} \right) - K, 0 \right) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$ (20)

![Figure 3: The underlying pdf $F_{S_t}(S)$ is mapped through the value function $\varphi(S)$ to match the triangular distribution.](image)

5 Results

In this section we provide a practical example of how managerial expectations of cash flows can be utilized to estimate the real option project value. Table 1 presents a contrived example of expected cash flows. A project launch cost of $450 will be required at the end of year 2 after which we have pessimistic, most likely and optimistic expected cash flows in years 3 to 9. The questions facing managers at time $t = 0$ is whether a small investment should be made now to have the option to invest in year 2 and realize the future cash flows.

It should be emphasized that the two methods of evaluations presented above are not directly consistent with each other, in that the Fitting method assumes that the cash flow process is correlated to a tradable asset, while the Matching method assumes that the cash flows are derived from a non-tradable indicator which is correlated to a traded asset. For this example, we assume that
Table 1: Managerial Expectations of Project Cash Flows.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimistic</td>
<td>0</td>
<td>0</td>
<td>80</td>
<td>120</td>
<td>150</td>
<td>180</td>
<td>200</td>
<td>220</td>
<td>250</td>
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<tr>
<td>Most likely</td>
<td>0</td>
<td>0</td>
<td>50</td>
<td>70</td>
<td>75</td>
<td>80</td>
<td>90</td>
<td>100</td>
<td>110</td>
</tr>
<tr>
<td>Pessimistic</td>
<td>0</td>
<td>0</td>
<td>20</td>
<td>25</td>
<td>25</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Investment</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>450</td>
</tr>
</tbody>
</table>

The risk-free rate, $r$, is 3%, the expected market growth, $\nu$, is 8% and the market volatility, $\eta$, is 10%. Due to the minimum martingale measure formulation for the Matching method, the results are independent of the expected market indicator parameters.

The fitted parameters of the Fitting method, $\mu_i$ and $\sigma_i$, are given in Table 2. The triangular cash flow PDF and CDF, and the corresponding fitted profiles, for $t = 6$ years are shown in Figure 4. As can be seen, the fitted distribution does not match the triangular distribution exactly. This discrepancy is avoided when the Matching method is implemented. The present value of the cash flows (not including the investment option of $450) are plotted in Figure 5. The discrepancy in the present value calculations is due to the imprecise fit of the Fitting Method.

Table 2: Fitted Parameters.

<table>
<thead>
<tr>
<th>Fitted Parameters</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_i$</td>
<td>0.00</td>
<td>0.36</td>
<td>0.16</td>
<td>0.10</td>
<td>0.09</td>
<td>0.09</td>
<td>0.11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_i$</td>
<td>0.21</td>
<td>0.19</td>
<td>0.22</td>
<td>0.34</td>
<td>0.19</td>
<td>0.19</td>
<td>0.22</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4: Probability density functions and cumulative density functions for the expected and fitted cash flows for the 6th year.

The option values for varying values of $\rho$ using both the Fitting and Matching methods are
plotted in Figure 6. As expected, the value of the option increases in both cases as $\rho$ decreases. The results show a slight difference in the value calculated for each method for lower correlations, however, the matching method shows zero option value at $\rho \geq 0.6$. The reason for this is that at higher market correlations, the market indicator can not reach a significantly high enough value in two years to make the project viable. Arguably, for the Fitting method, the managers would be expected to provide an estimate for $\rho$, while the Matching method can rely on historical data to estimate this parameter.

![Graph](image)

Figure 5: Present value of the cash flows using the Fitting and Matching methods.

6 Concluding Remarks

Fuzzy numbers provide an intuitive method for managers to forecast project future payoffs and have gained some popularity in practical use in real options analysis. The methods discussed here build on previously published fuzzy real options valuation techniques, but add financial rigour consistent with financial theory. Both methods properly account for the inherent correlation of the expected values of payoffs as time evolves to maturity and the methods allow for the estimated cash flows to be appropriately accounted for with respect to market versus idiosyncratic risk. Neither of the methods rely on simulation which makes them more tractable for optimization and can be easily implemented in a spreadsheet. We are especially confident with the tractability of the Matching method for it only requires 1) managements’ estimates of future cash flows, 2) a market sector
Figure 6: Option values calculated using the Fitting and Matching methods.

indicator process (for example historical market size estimates can be used to establish a volatility value for the indicator process and market forecasts can be used to estimate its growth) and 3) a traded market index or stock that is highly correlated to the market sector indicator.

References


