Defaultable bonds under imprecise information

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Abstract

This article develops a computational method to implement the effect of imperfect information on the value of defaultable bonds. A fuzzy modeling is adopted and the numerical experiments show that an imprecise value of the stochastic underlying asset and/or the barrier triggering the default have material impact on the qualitative shape of the term structures of credit spreads.

Keywords: Stochastic process, fuzzy sets, defaultable bonds, incomplete information.

1. Introduction

It is recognized that several factors have contributed to trigger the recent financial crisis: the search for yield enhancement, agency problems, lax underwriting standards, failure by the rating agencies to identify a changing environment, poor risk management by financial institutions, lack of transparency, the limitation of extant valuation models and the failure of regulators to understand the implications of the changing environment for the financial system (see Crouhy et al. (2008)). In this paper we focus on the insufficiency of disclosure as a determinant of the value of credit risk-sensitive financial instruments. A common view is that “disclosure remains the sole paradigm for remedying the information asymmetry between originators and investors”. In order to understand the role of asymmetric information in Finance one has to test the relevance of the investors’ imprecise information upon the performance of single name credit risk models. Such an issue has been the object of a recent stream of financial literature that, starting from Duffie and Lando (2001), has developed into the incomplete information approach. It has been shown that the imperfect information of the investors is responsible for the undervaluation of high credit standing obligors. In this paper we show that, in modeling imprecise information, the fuzzy approach can play a role and can provide a computationally simpler alternative to the incomplete information methodology. This article extends the results in Agliardi and Agliardi (2009) where this point of view has been adopted employing the classical Merton (1974) model. In this paper we build on the Black - Cox model (1976) which is the seminal work of the first-passage approach in the credit risk literature. The main idea of this stream of literature is to model explicitly the mechanism behind default events by assuming that financial distress is triggered when the firm asset value falls below a pre-specified barrier. The interpretation is that the assets of the firm are insufficient to meet
payments on debt, where debt includes the nominal repayment of the bond (like in Merton (1974))
together with a multitude of other debt components which cannot be modelled in detail and are
therefore treated on an aggregate level as a "default threshold". (See the discussion in Agliardi
(2010)). Clearly the value of such a threshold is affected by imprecise information, which fuels
motivation to our present investigation.

An important measure of bond defaultability is the size of its credit spread, that is the excess yield
of a defaultable bond on the yield of a similar credit risk-free bond. Spreads forecasts by a classical
first-passage model are higher, and thus more realistic, than in the original Merton model, but still
they do not fit perfectly with the empirically observed ones (see Eom et al. (2004)). Indeed, one
major problem with structural models is the under prediction of short-term credit spreads: according
to these models, the credit spread declines rapidly to zero as the residual life of the bond tends to
zero, that is, a junk bond is no riskier than a Treasury bond over a short term. This result is a
consequence of the assumption that the default event is announced by the observation of the firm's
value coming close to the default boundary and thus it occurs with some warning, since the firm's
value is modeled as a continuous simple path process. In reality default often hits the market by
surprise. A successful attempt to overcome these drawbacks, while maintaining the link between the
default event and the fundamental variable of the firm, has come recently to the light and is based
on the concept of incomplete information. The first model of this kind is introduced in Duffie and
Lando (2001), where the first-passage time definition of default is retained but investors are
assumed to observe the firm's fundamental variable only imperfectly. By introducing noise into the
market information set, this mathematical approach makes the default time inaccessible to investors
and results in non zero short-term credit spreads. An interpretation of incomplete information is that
accounting reports either purposefully or inadvertently obscures the market knowledge of the firm's
asset value and of the liabilities. Recent corporate accounting scandals can be framed in the first
scenario. In any case, the market cannot observe the true value, which is known only by the
management of the firm, and the publicly available information provides only an imprecise measure
of the true value. Our aim is to show the effect of imprecise information on the bond price and
credit spread, when a fuzzy framework is employed. We have chosen to adopt the classical Black-
Cox model and rule out the technicalities of the subsequent more recent literature on corporate
bonds, because our goal is to point out the impact of the imprecise information assumption. The
main focus is on the numerical computation of the average credit spreads for different credit ratings
and the determination of their term structures. We show that the fuzziness of the firm total asset
value and/or the default threshold helps to attenuate the drawbacks of the structural approach and
thus improves the performance of the model. Moreover, the shape of the term structure is in keeping
with the recent literature on incomplete information (see Giesecke and Goldberg (2004)). Finally,
we point out that the results presented in this paper provide a generalization of Agliardi and
Agliardi (2009), in that we investigate the effect of imprecise knowledge both on the asset value
and on the aggregate debt value which is modeled as a triggering barrier.

2. Preliminary notation

Since imprecise information on the financial values will be modeled in a fuzzy framework, in this
section we outline the basic fuzzy-stochastic notation. For a comprehensive presentation of fuzzy
theory the main references are Zadeh (1965) and Dubois and Prade (2000). A fuzzy number is a
fuzzy set (depicted with tilde) of the real line $\mathbb{R}$, which is commonly defined by a normal, upper-
semicontinuous, fuzzy convex membership function $\mu: \mathbb{R} \rightarrow [0,1]$ of compact support. The $\gamma$-cut of a
fuzzy number is given by $\bar{\mu}_\gamma = \{x \in \mathbb{R} : \mu(x) \geq \gamma\}$, $\gamma \in [0,1]$. The closed intervals are written as
$\bar{\mu}_\gamma = [\bar{\mu}_\gamma^-, \bar{\mu}_\gamma^+]$ for $\gamma \in [0,1]$. The arithmetic operations on two fuzzy numbers can be defined in the
standard way, in terms of the $\gamma$-cuts for $\gamma \in [0,1]$. In particular, for fuzzy numbers $\bar{\mu}$ and $\bar{\nu}$, the
addition and subtraction and the scalar multiplication \( a \tilde{\mu} \), where \( a \geq 0 \), are fuzzy numbers as follows:
\[
(\tilde{\mu} + \tilde{v})_\gamma = [\tilde{\mu}^- + \tilde{v}^-, \tilde{\mu}^+ + \tilde{v}^+],
\]
\[
(\tilde{\mu} - \tilde{v})_\gamma = [\tilde{\mu}^- - \tilde{v}^-, \tilde{\mu}^+ - \tilde{v}^+],
\]
\[
(a \tilde{\mu})_\gamma = [a \tilde{\mu}^-, a \tilde{\mu}^+].
\]
Moreover, multiplication and division between two fuzzy numbers \( \tilde{\mu} \) and \( \tilde{v} \) are given by:
\[
(\tilde{\mu} \tilde{v})_\gamma = [(\tilde{\mu} \tilde{v})^-_\gamma, (\tilde{\mu} \tilde{v})^+_\gamma],
\]
where
\[
(\tilde{\mu} \tilde{v})^-_\gamma = \min(\tilde{\mu}^-, \tilde{v}^-)^-, (\tilde{\mu}^+, \tilde{v}^-)^+, (\tilde{\mu}^-, \tilde{v}^+)^-, (\tilde{\mu}^+, \tilde{v}^+)^+,
\]
\[
(\tilde{\mu} \tilde{v})^+_\gamma = \max(\tilde{\mu}^-, \tilde{v}^-)^-, (\tilde{\mu}^+, \tilde{v}^-)^+, (\tilde{\mu}^-, \tilde{v}^+)^-, (\tilde{\mu}^+, \tilde{v}^+)^+.
\]
and
\[
(\tilde{\mu} / \tilde{v})_\gamma = [(\tilde{\mu} / \tilde{v})^-_\gamma, (\tilde{\mu} / \tilde{v})^+_\gamma] \quad (\text{with } \tilde{v} > 0 \text{ or } \tilde{v} < 0)
\]
where
\[
(\tilde{\mu} / \tilde{v})^-_\gamma = \min(\tilde{\mu}^-, \tilde{v}^-)^-, (\tilde{\mu}^+, \tilde{v}^-)^+, (\tilde{\mu}^-, \tilde{v}^+)^-, (\tilde{\mu}^+, \tilde{v}^+)^+,
\]
\[
(\tilde{\mu} / \tilde{v})^+_\gamma = \max(\tilde{\mu}^-, \tilde{v}^-)^-, (\tilde{\mu}^+, \tilde{v}^-)^+, (\tilde{\mu}^-, \tilde{v}^+)^-, (\tilde{\mu}^+, \tilde{v}^+)^+.
\]

A fuzzy-number-valued map \( \tilde{X} \) is called a fuzzy random variable if \( \{(\omega, x) \in \Omega \times \mathbb{R}; \tilde{X}(\omega)(x) \geq \gamma\} \) is measurable for all \( \gamma \in [0, 1] \). In what follows fuzzy stochastic processes \( \tilde{X}_t \) are employed, that is, \( \tilde{X}_t \) are fuzzy random variable for every \( t \in [0, T] \).

### 3. Bond valuation under imprecise information

This section is devoted to the introduction of a financial model for corporate bond valuation. Since we build on the classical Black-Cox model, we recall briefly this model. Let \( V \) denote the value of the firm assets, which is modeled as a geometric stochastic process. It is assumed that the firm has a zero-coupon bond outstanding maturing at date \( T \) and whose face value is \( B \). The short-term risk-free interest rate is \( r \). The possibility that the bondholders have the right to force reorganization or bankruptcy, when the firm performs poorly, is modeled as follows. If the firm value falls below a specified level, which is a function of time, then the bondholders force the firm into bankruptcy and receive the total firm value. We assume that the default threshold at time \( t \) is of the form \( He^{-r(T-t)} \).

Moreover the liquidation of the firm occurs at the maturity date whenever \( V < B \), that is, bondholders cannot be paid in full. The current value of the bond, \( P(V, t) \), is found by solving the partial differential equation:

\[
\partial_t P + \frac{\sigma^2 V^2}{2} \partial^2_{\sigma^2} P + rV \partial_V P = rP
\]

with final condition \( P(V,T)=\min(V,B) \) and boundary condition \( P(He^{-r(T-t)}, t)=He^{-r(T-t)} \).

Then the expression for \( P(V, t) \) is given by:
\[ B e^{-rT} N(z_1^{-}) + VN(-z_1^{-}) + H e^{-rT} N(z_2^{+}) - \frac{BV}{H} N(z_3^{+}) + (H e^{-rT} - V) N(z_3) \]  

where \( \tau = T-t \),  
\[ z_1^{+} = (\ln \left( \frac{V}{B} + rT \pm \frac{\sigma^2 \tau}{2} \right) / (\sigma \sqrt{\tau}) , \quad z_2^{+} = (\ln \left( \frac{H^2}{BV} - rT \pm \frac{\sigma^2 \tau}{2} \right) / (\sigma \sqrt{\tau}) , \]  
\[ z_3^{+} = (\ln \left( \frac{V}{H} + rT - \frac{\sigma^2 \tau}{2} \right) / (\sigma \sqrt{\tau}) \]  
and \( N \) is the standard normal distribution function.

Like all the classical structural models of bond pricing this model is based on complete information: investors have an accurate picture of the firm’s asset value and liabilities. In this paper we depart from this unrealistic assumption and take the alternative view that investors observe the true value of the firm only imperfectly. Then the analysis is extended to the imperfect observation of the default threshold \( H \). To this purpose we suppose that \( \tilde{V}_t \) is a fuzzy stochastic process, which is defined as follows:

\[ \tilde{V}_t(\omega) = 1+(-xV_t(\omega)) / \alpha^- V_t(\omega) \quad \text{if} \quad V_t(\omega)(1-\alpha^-) \leq x \leq V_t(\omega) \]  
\[ \tilde{V}_t(\omega) = 1+(-xV_t(\omega)) / \alpha^+ V_t(\omega) \quad \text{if} \quad V_t(\omega) \leq x \leq V_t(\omega)(1+\alpha^+) \]  
and \( = 0 \) otherwise,

with \( 0 < \alpha^+ \leq \alpha^- < 1 \). That is, the fuzzy random variable \( \tilde{V}_t \) is of the triangular type, with centre \( V_t(\omega) \), left-width \( \alpha^- V_t(\omega) \) and right-width \( \alpha^+ V_t(\omega) \). Note that our definition follows Yoshida (2003). The assumption \( \alpha^+ \leq \alpha^- \) is related to the investors’ subjective belief about the reliability of the accounting data of the firm: they tend to shift the known firm value to the left, because the published value of \( V \) is more likely to be overstated than understated as misreporting or fraudulent behavior are a possibility. Observe that the fuzziness in the process increases as \( \alpha^+ \) become bigger.

The \( \gamma \)-cuts of \( \tilde{V}_t(\omega) \) are:

\[ [V_t(\omega)(1-1-\gamma)\alpha^-), V_t(\omega)(1+(1-\gamma)\alpha^+)] \]  

Let us now state the fuzzy generalization of (1), which is obtained in view of the monotonicity of \( P(V,t) \) with respect to \( V \), for reasonable parameter values, and the fuzzy arithmetic.

**Proposition.** The rational price of the defaulatable bond considered above is given by:

\[ \tilde{P}_y(V,t) = B e^{-rT} N(z_1^{-}) + V(1 \pm (1-\gamma)\alpha^+) N(z_1^{+}) + H e^{-rT} N(z_2^{+}) - \frac{BV}{H}(1 \pm (1-\gamma)\alpha^+) N(z_3^{+}) \]  

where the \( z^\pm \)'s are obtained from the corresponding argument in (1) replacing \( V \) with \( V(1 \pm (1-\gamma)\alpha^+) \) and \( \gamma \in [0,1] \).

4. Numerical experiments

In this section the model is implemented to compute the bond price and credit spread. Computation and plots are obtained employing Mathematica. As the base case, we take a triangle-type shape function for the fuzzy firm value and assume an asymmetry that accounts for the investors’ subjective beliefs about the reliability of the accounting transparency of the firm. (See Figure 1).
The several recent accounting scandals that have shocked the business world suggest that misreporting more likely overstate the true asset value and thus the triangle is asymmetric to the left, in that it represents the investors’ estimate of the true value. Figure 2 depicts the resulting function for the fuzzy bond price in a base case with \( V=400 \), \( B=100 \), \( H=72 \) and maturity three years. The choice of the default threshold as a 72% of the face value of the debt is motivated by the empirical investigation. Here the risk-free rate is fixed at 1%.

Finally the model is calibrated to be consistent with historical data of different rating categories. The analysis is restricted to A- and B-rated firms to study how the model works on speculative and investment grade bonds. The values of the quasi-leverage ratio and the volatility are taken from Huang and Huang (2003). Our aim is to investigate the impact of a fuzzy framework on the term structure of credit spreads. The T-maturity of credit spread is defined as usual:

\[
s_{T}(V) = -\frac{1}{T} \ln(P(V,0)) - r
\]

In Figures 3 and 4 the term structure of the credit spreads is plotted for the two credit qualities considered above. The dashed lines represent \((\tilde{\delta}_V,\tilde{\delta}_B)\), that is, the credit spreads of the most pessimistic and the most optimistic investors. The average of the two extreme values is represented by the solid thin line, while the solid thick line plots the crisp value. Note that the recognized shortcoming of structural models - to be unable to predict sufficiently large spreads on short-term - seems to be mitigated in a fuzzy framework, especially for low credit quality.

Indeed, the average short spread of the poorer-quality bonds are significantly positive, in keeping with empirical evidence and with the literature about the imperfect information modeling (Duffie and Lando (2001), Giesecke and Goldberg (2004)).
Now we parallel the analysis in Giesecke and Goldberg (2004) and assume that the default threshold is not revealed to investors, that is, $H$ is a fuzzy number. The more interesting situation occurs when its triangular shape triangle is asymmetric to the right. Note that, since (1) is a decreasing function of $H$, the analogous of Proposition 1 holds with an expression of the form $H(1 \pm (1 - \gamma') \beta^7)$ replacing $H$, where $\gamma' \in [0,1]$. Figure 5 shows the result of a simulation in a base case. The credit spread is significantly bounded away from 0 for short maturities.

![Credit spread (fuzzy H)](image)

Figure 5

5. Concluding remarks

The main criticism which is usually cast on the structural approach is to predict too low credit spreads at short maturities. A first successful attempt to get empirically plausible credit spreads in a structural framework is Zhou (1997), where a jump-diffusion process for the firm value is adopted. This paper, however, retains the view that the assets of the firm are perfectly observable and, as Duffie and Lando (2001) points out, it “offers no role in the estimation of default risk for some potentially useful explanatory variables”. The issue of properly modeling the informational set of investors in the secondary bond market to draw all the consequences of the incomplete information assumption was developed by Duffie and Lando (2001) for the first time. It results in a successful integration of structural and reduced-form models. While Duffie and Lando (2001) is based on Leland and Toft, where the default threshold is endogenous, in this paper we adopt the Black-Cox model and thus we are able to study also the effect of an imperfect observation of the default barrier. A similar approach is taken in Giesecke-Goldberg (2004) where the Black-Cox model is generalized, in that the default barrier is a random variable that follows a scaled beta distribution. In our paper both the value of the firm assets and the default threshold are modeled as fuzzy numbers and the assumptions concern the membership function. As we emphasized in Section 4, both Giesecke and Goldberg (2004) and our approach are able to generate qualitatively more realistic short-term credit spreads than the traditional structural approach. As Agliardi and Agliardi (2009) shows, there are also further implications about corporate bonds which are consistent with empirical evidence and can be obtained in a fuzzy framework but are precluded in a classical structural modeling. A fuzzy approach allows us to examine the effects of different risk factors separately, while in the incomplete information approach the emphasis lies on the updating of the informational set.

The aim of this work is to present an alternative model of the imprecise information that is available to investors in practice. The combination of a structural approach, even a parsimonious one, with a fuzzy framework allows to model explicitly the mechanism behind default while introducing more flexibility than in traditional modeling. We have shown that such an approach has the flexibility for capturing the basic features of the observed credit spreads of the bonds. This opens up a further scope of research in credit risk analysis.
6. References


Huang J, Huang M (2003). How much of the Corporate -Treasury yield spread is due to credit risk?, W.P. Stanford University


