An algorithm to find the best approximate solutions for a particular fuzzy relational equations with max-product composition

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Abstract

Fuzzy relational equations have played an important role in fuzzy modeling and have applied to many practical problems. Most theoretical results of fuzzy relational equations based on a premise that the solution set is nonempty. However, it is commonly seen that the case of fuzzy relational equations is inconsistent. The inconsistent fuzzy relational equation is so-called “inverse fuzzy relation” problem. Finding the approximate solution for inverse fuzzy relation problem has been investigated by several authors. The proposed algorithm for solving the inverse fuzzy relation problem usually based on the genetic algorithm (GA) or heuristic algorithm. However, these algorithms are expected to yield good results in most cases but are not guaranteed to yield the best approximate solution. To provide a precisely solution procedure, an algorithm to find the best approximate solution of the inverse fuzzy relation problem included the right hand side vector \( b = (1 \ 0 \ \cdots \ 0) \) with max-product composition is presented in this study. Numerical examples are also provided to illustrate how the solution algorithm can be applied to find the best approximate solution for the studied problem.

Keywords: Fuzzy relational equations, Max-product composition, The best approximate solution

1. Introduction

In this paper, the following fuzzy relational equations are considered:

\[ x \circ A = b \]  

(1)

where \( x = (x_i)_{1 \times m} \) is an \( m \)-dimensional vector, \( A = (a_{ij})_{m \times n} \) is an nonnegative matrix, \( b = (b_j)_{1 \times n} \) is an \( n \)-dimensional vector, \( a_{ij}, b_j, x_i \in [0,1] \) for each \( i, 1 \leq i \leq m \) and for each \( j, 1 \leq j \leq n \). The operation “\( \circ \)” in (1) represents the max-product composition.

Let \( X(A, b) := \{ x \in [0,1]^m | x \circ A = b \} \) denote the solution set of (1). The solution set of (1) is empty, \( X(A, b) = \emptyset \), then \( x = (x_i)_{1 \times m} \) is called the approximate solution of (1). Let \( x = (x_i)_{1 \times m} \) be an approximate solution of (1) and satisfy \( x \circ A = b' \) such that the following \( D(x) \) has the smallest distance among all possible approximate solutions, then \( x \) is called the
best approximate solution of (1),

\[ D(x) = \sum_{j=1}^{n} |b_j - b'_j|, \]

where \(|b_j - b'_j|\) denotes the Hamming distance norm of elements \(b_j\) and \(b'_j\).

Let \(\mathcal{I} = \{1, 2, \cdots, m\}\) and \(\mathcal{J} = \{1, 2, \cdots, n\}\) be two index sets. Solving a fuzzy relational equation for (1) with max-product composition is to find a set of solution vectors \(x = (x_i)_{i \in \mathcal{I}}\) such that

\[ \max_{i \in \mathcal{I}} \{a_{ij}x_i\} = b_j, \forall j \in \mathcal{J}. \]

Investigating the best approximate solution of fuzzy relational equation problem with max-product composition in (1) is to find a set of vectors \(x = (x_i)_{i \in \mathcal{I}}\) such that the following measured norm \(D(x)\) has the smallest distance.

\[ D(x) = \sum_{j=1}^{n} |b_j - b'_j| = \sum_{j=1}^{n} |b_j - \max_{i \in \mathcal{I}} \{a_{ij}x_i\}| \quad (2) \]

It is easy to note that \(D(x) = 0\) stands for the fuzzy relational equations in (1) is consistent. On the other hand, if the fuzzy relational equations in (1) is inconsistent, then \(D(x) > 0\).

Finding solution of Eq. (1) belongs to the topic of fuzzy relational equation problems, which have played an important role in fuzzy modeling and have been studied by many researches. There are many different kinds of such equations, each based on a specific composition of fuzzy relations. The first paper of fuzzy relational equations was due to Sanchez [1], where the man-min composition was considered. Later on, Pedrycz [4] studied the max-product fuzzy relational equations. Since the commonly seen max-min and max-product compositions are special cases of the max-triangular-norm (max-t-norm) composition, many discussions [14,16] have extended to the resolution of fuzzy relational equations with max-t-norm composition. It now becomes well known that the solution set of a consistent finite system of fuzzy max-t-norm relational equations can be completely determined by a maximum solution and a finite number of minimal solutions. Generally, the maximum solution in solution set can be easily computed by an analytic formula. However, as shown in Markovskii [12], solving max-product fuzzy relational equations to find all the minimal solutions is closely related to the set covering problem, which is an NP-hard problem. Luoh et al. [11] proposed a matrix-pattern-based computer algorithm to solve for max-min and max-product fuzzy relational equations. Guu and Wu [9] provided a necessary condition for the max-product fuzzy relational equations. This necessary condition provides that, for a minimal solution, each of its components is either 0 or the corresponding component’s value of the maximum solution. Various approaches developed to detect the minimal solutions for fuzzy max-t-norm relational equations with a specific composition can may be found in [13,14,17].

Generally, the theory of fuzzy relational equations mostly relies on the assumption that the solution set is not empty. However, this assumption is often not the case in practical applications. Hence, a broader theory that would allow us to determine an adequate approximate solution for inverse fuzzy relation problem is necessary to study. Study on the approximate solutions of inverse fuzzy relation problem can be found in the literature. The work of Pedrycz [2] is the first paper to provide an approximate solution for fuzzy relational equations when the solution set is empty. This work applied a modified Newton method for which \(x \circ A\) has a minimal distance from \(b\). Since then another method proposed by Gottwald and Pedrycz [5] was based on a solvability index. According to this method, one given equation is appropriately modified to
increase the solvability index until an equation is obtained that is solvable. As those arguments in the work of Klir and Yuan [7] this method is rather inefficient and leads usually to a trivial solution.

By means of the equality index proposed by Gottwald [3], Klir and Yuan [7] introduced a goodness measure of the performance of approximate solutions and derived a lower bound and an upper bound of solvability of stems of fuzzy relation equations. Investigating the approximate solution of inverse fuzzy relation problem with max-min composition, Pedrycz [6] proposed an algorithm to modify the original vector \( b \) of problem (1) in a way leading to its genuine solution or at least an approximate one. The proposed algorithm used the minimum distortion principle which measured by the Hamming distance as the main criterion to obtain one modified vector to get a solution of the given equation. The approximate solution obtained by this proposed algorithm did not show to be the best approximate solution of (1) although the Hamming distance as (2) was considered.

Several related issues of inverse fuzzy relation problem still discussed recently. Louh [15] extended problem (1) to the case \( X \circ A = B \), where \( X \) is an unknown \( s \times m \) matrix, \( A \) and \( B \) are two known \( m \times n \) and \( s \times n \) matrices, respectively. Then the real-valued genetic algorithm (RVGA) is proposed to solve an approximate solution for the max-product fuzzy relational equation. Saha and Konar [10] considered the following case

\[
Q \circ A = I
\]

(3)

where \( Q \) is an unknown \( n \times m \) matrix, \( A \) is a known \( m \times n \) matrix, \( I \) denotes the \( n \times n \) identity matrix, and the man-min composition was considered. They defined a heuristic function to reduce the search space for finding the approximate solution of problem (3), where \( D(x) \) in (2) was measured by a Euclidean distance norm and employed it in classical abductive reasoning problems. Recently, Chakrabarty et al. [18] proposed an efficient algorithm to obtain the best approximate solution which measured by a Hamming distance norm for problem (3). The time-complexity of the proposed algorithm is \( O(n^2) \). However, De Baets [8] showed that the problem (3) can be decomposed into a set of (1). Namely, the problem (3) can be solved \( n \) times, repeatedly, by using the method of solving (1). Motivated by the work of Chakrabarty et al. [18], some theoretical results are proposed for solving problem (1) with the right hand side vector \( b = (b_j)_{1 \times n} = (1 \ 0 \ \cdots \ 0) \). An algorithm for finding the best approximate solution to the max-product inverse fuzzy relation problem is proposed. Numerical examples are also provided to illustrate how the solution algorithm can be applied to find the best approximate solution for the studied problem.

2. Preliminary properties

In this section, some theoretical results are presented for the problem (1) with the right hand side vector \( b = (b_j)_{1 \times n} = (1 \ 0 \ \cdots \ 0) \) when the solution set is empty, \( X(A, b) = \emptyset \). For convenience, the studied problem can be detailed as follows:

\[
x \circ A = (x_1 \ x_2 \ \cdots \ x_m) \circ \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \cdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} = (b_1 \ b_2 \ \cdots \ b_n) = (1 \ 0 \ \cdots \ 0).
\]
When the symbol ‘‘⊙’’ denotes the max-product composition, this problem can be transferred into the following model.

\[
\begin{align*}
\max \{a_{11}x_1, a_{21}x_2, \ldots, a_{m1}x_m\} &= 1 \\
\max \{a_{12}x_1, a_{22}x_2, \ldots, a_{m2}x_m\} &= 0 \\
\vdots & \vdots \\
\max \{a_{1n}x_1, a_{2n}x_2, \ldots, a_{mn}x_m\} &= 0.
\end{align*}
\]

(4)

It is easy to note that \(x = (x_i)_{i \in I}\) contains \(x_i = 1\) and \(x_k = 0\) for all \(k \in I\) and \(k \neq i\) is a solution of problem (4) if \(a_{i1} = 1\) and \(a_{ij} = 0\) for all \(j \in \{2, 3, \ldots, n\}\) are given. However, the following situations show the problem (4) no solutions.

1. If \(a_{i1} < 1\), \(\forall i \in I\), then we have \(\max \{a_{i1}x_1, a_{21}x_2, \ldots, a_{m1}x_m\} < 1\). This implies the first equation of (4) can not be satisfied and \(X(A, b) = \emptyset\).

2. If there exists \(a_{ij} > 0, j \neq 1, j \in J\), then \(x_i\) must be qual to 0; otherwise, \(x_i = 0\) implies \(\max \{a_{1j}x_1, a_{2j}x_2, \ldots, a_{m1}x_m\} > 0\) and \(X(A, b) = \emptyset\).

3. Obviously, if there does not exist \(a_{i1} = 1\) and \(a_{ij} = 0\), for all \(j \in \{2, 3, \ldots, n\}\) in the \(i\)th row of matrix \(A\), then no solutions exist for (4).

Based on the above situations, fuzzy relational equations in (4) are commonly inconsistent. Assume that the solution set of problem (4) is empty, then how to find the best approximate solution \(x\) to satisfy \(x \circ A = b'\) such that the measured Hamming norm \(D(x)\) in (2) has the smallest distance becomes to the main work. Notably, for any approximate solution \(x = (x_i)_{i \in I}\) of (4), \(D(x)\) in (2) can be extended to as follows:

\[
D(x) = \sum_{j=1}^{n} |b_j - b'_j| = \sum_{j=1}^{n} |b_j - \max_{i \in I} \{\min(a_{ij}x_i)\}|
\]

\[
= |1 - \max_{i \in I} \{a_{i1}x_i\}| + \sum_{j=2}^{n} |0 - \max_{i \in I} \{a_{ij}x_i\}|
\]

\[
= 1 - \max_{i \in I} \{a_{i1}x_i\} + \sum_{j=2}^{n} \max_{i \in I} \{a_{ij}x_i\}
\]

(5)

It is clear that \(x = (x_i)_{i \in I} = 0\) is an approximate solution of (4). Given this approximate solution \(x = 0\) to Eq. (5), we can get \(D(x) = 1\) and the following lemma.

**Lemma 1.** For any approximate solution \(x = (x_i)_{i \in I}\) of problem (4), if there exists \(D(x) > 1\), then \(x\) is not a best approximate solution of (4).

From Lemma 1, it shows that the sufficient condition for the existence of best approximate solutions to problem (4) is \(D(x) \leq 1\). On the other hand, if the problem (4) exists best approximate solutions, then \(\max_{i \in I} \{a_{i1}x_i\} \geq \sum_{j=2}^{n} \max_{i \in I} \{a_{ij}x_i\}\) must hold. To further analyze the property of \(D(x)\), the following operation can easily derive from the Eq. (5):

\[
D(x) = 1 - \max_{i \in I} \{a_{i1}x_i\} + \sum_{j=2}^{n} \max_{i \in I} \{a_{ij}x_i\}
\]

\[
= 1 - \max \{a_{11}x_1, a_{21}x_2, \ldots, a_{m1}x_m\} + \max \{a_{12}x_1, a_{22}x_2, \ldots, a_{m2}x_m\} + \cdots
\]

\[
+ \max \{a_{1n}x_1, a_{2n}x_2, \ldots, a_{mn}x_m\} + \cdots + \max \{a_{1n}x_1, a_{2n}x_2, \ldots, a_{mn}x_m\}
\]

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Looking closely for above equations, it is easy to note that all variables can be separated to compute the value $D(x)$. Hence, one component $x_i$ can be selected from approximate solution $x = (x_i)_{i \in \mathcal{I}}$ of problem (4) to yield the corresponding value that denoted by $E(x_i)$ as follows:

$$E(x_i) = 1 - a_{i1}x_i + \sum_{j=2}^{n} a_{ij}x_i.$$ 

**Theorem 1.** For any approximate solution $x = (x_i)_{i \in \mathcal{I}}$ of problem (4),
(i) there exists at least one $i \in \mathcal{I}$ such that $D(x) \geq E(x_i)$.
(ii) if $x^*$ is the best approximate solution of problem (4), then $D(x^*) = \min_{i \in \mathcal{I}}\{E(x_i)\}$.

**Proof:**
(i). Since $\sum_{j=2}^{n} \max_{i \in \mathcal{I}}\{a_{ij}x_i\} \geq \sum_{j=2}^{n} (a_{ij}x_i), \forall i \in \mathcal{I}$, this yields

$$D(x) = 1 - \max_{i \in \mathcal{I}}\{a_{i1}x_i\} + \sum_{j=2}^{n} \max_{i \in \mathcal{I}}\{a_{ij}x_i\} \geq 1 - \max_{i \in \mathcal{I}}\{a_{i1}x_i\} + \sum_{j=2}^{n} (a_{ij}x_i), \forall i \in \mathcal{I}.$$ 

Furthermore, $\max_{i \in \mathcal{I}}\{a_{i1}x_i\}$ must exist the largest value, thus we can let the largest value $\max_{i \in \mathcal{I}}\{a_{i1}x_i\} = a_{i1}x_i$ and obtain

$$D(x) \geq 1 - \max_{i \in \mathcal{I}}\{a_{i1}x_i\} + \sum_{j=2}^{n} a_{ij}x_i = 1 - a_{i1}x_i + \sum_{j=2}^{n} a_{ij}x_i = E(x_i).$$

(ii). The approximate solution $x = (x_i)_{i \in \mathcal{I}}$ of problem (4) contains different number of elements are considered as follows:
(1). Only one element in $x$ is greater than or equal to 0 and the others elements are equal to 0. Let $x = (0, \cdots, 0, x_i, 0, \cdots, 0)$ contain $x_i \geq 0$ and $x_k = 0$ for all $k \in \mathcal{I}$ and $k \neq i$, then $D(x) = 1 - a_{i1}x_i + \sum_{j=2}^{n} a_{ij}x_i = E(x_i)$. Hence, if $x^* = (0, \cdots, 0, x_i^*, 0, \cdots, 0)$ is the best approximate solution of problem (4), then $D(x^*) = \min_{i \in \mathcal{I}}\{E(x_i)\}$.

(2). Two elements in $x$ are greater than or equal to 0 and the others elements are equal to 0. Let $x = (0, \cdots, 0, x_i, x_s, 0, \cdots, 0)$ contain $x_i \geq 0, x_s \geq 0$ and $x_k = 0$ for all $k \in \mathcal{I}$ and $k \neq i, k \neq s$, then $D(x) = 1 - \max\{a_{i1}x_i, a_{s1}x_s\} + \sum_{j=2}^{n} \max\{a_{ij}x_i, a_{sj}x_s\}$. To determine the value of $D(x)$, two cases for $\max\{a_{i1}x_i, a_{s1}x_s\}$ are needed to discuss.

**Case 1.** $a_{i1}x_i \geq a_{s1}x_s$. This case implies

$$D(x) = 1 - a_{i1}x_i + \sum_{j=2}^{n} \max\{a_{ij}x_i, a_{sj}x_s\} \geq 1 - a_{i1}x_i + \sum_{j=2}^{n} a_{ij}x_i = E(x_i).$$

**Case 2.** $a_{s1}x_s \geq a_{i1}x_i$. This case implies

$$D(x) = 1 - a_{s1}x_s + \sum_{j=2}^{n} \max\{a_{ij}x_i, a_{sj}x_s\} \geq 1 - a_{s1}x_s + \sum_{j=2}^{n} a_{sj}x_s = E(x_s).$$

Depending on Case (1) and Case (2), we have $D(x) \geq \min\{E(x_i), E(x_s)\}$. Hence, if $x^* = (0, \cdots, 0, x_i^*, x_s^*, 0, \cdots, 0)$ is the best approximate solution of problem (4), then $D(x^*) = \min_{i \in \mathcal{I}}\{E(x_i)\}$.

(3). The approximate solution $x = (x_i)_{i \in \mathcal{I}}$ contains different number of elements (more than two elements) are greater than or equal to 0 except for (1) and (2). The proof for (3) is similar to that for (2).

To summarize the situations (1), (2) and (3), we can obtain $D(x) \geq \min_{i \in \mathcal{I}}\{E(x_i)\}$ for any approximate solution $x = (x_i)_{i \in \mathcal{I}}$. Therefore, if $x^*$ is the best approximate solution of problem
(4), then \( D(x^*) = \min D(x) = \min_{i \in \mathcal{I}} \{E(x_i)\} \).

From Theorem 1, one can conclude that \( D(x) \geq E(x_i) \) must hold for at least one \( i \in \mathcal{I} \) in problem (4). Moreover, solving the best approximate solution of problem (4) is equivalent to finding the element \( x_i \) of \( x = (x_i)_{i \in \mathcal{I}} \) such that \( E(x_i) \) has the smallest value.

**Lemma 2.** For any approximate solution \( x = (x_i)_{i \in \mathcal{I}} \) of problem (4), if there exists \( a_{i_1} \leq \max_{j=\{2,3,\ldots,n\}} \{a_{ij}\} \) or \( a_{i_1} \leq \sum_{j=2}^n a_{ij} \), then \( E(x_{i_1}) \geq 1 \).

**Proof:** Since \( a_{i_1} \leq \max_{j=\{2,3,\ldots,n\}} \{a_{ij}\} \) or \( a_{i_1} \leq \sum_{j=2}^n a_{ij} \), we have \( a_{i_1} x_i \leq (\sum_{j=2}^n a_{ij}) x_i = \sum_{j=2}^n a_{ij} x_i \). Hence, \( E(x_{i_1}) = 1 - a_{i_1} x_i + \sum_{j=2}^n a_{ij} x_i \geq 1 \).

By Lemma 2, one can easily acquire that if \( a_{i_1} \leq \max_{j=\{2,3,\ldots,n\}} \{a_{ij}\} \) or \( a_{i_1} \leq \sum_{j=2}^n a_{ij} \), then the existence of the smallest value \( E(x_{i_1}) = 1 \) must be \( x_i = 0 \) or \( a_{i_1} = \sum_{j=2}^n a_{ij} \) with \( x_i \in [0,1] \).

**Lemma 3.** For any approximate solution \( x = (x_i)_{i \in \mathcal{I}} \) of problem (4), if there exists \( a_{i_1} \geq \sum_{j=2}^n a_{ij} \), then given \( x_i = 1 \) can get the smallest value \( E(x_{i_1}) = 1 - a_{i_1} + \sum_{j=2}^n a_{ij} \leq 1 \).

**Proof:** Follows from the definition of \( E(x_{i_1}) \), we have

\[
E(x_{i_1}) = 1 - a_{i_1} x_i + \sum_{j=2}^n a_{ij} x_i = 1 - [a_{i_1} - \sum_{j=2}^n a_{ij}] x_i.
\]

Since \( a_{i_1} \geq \sum_{j=2}^n a_{ij} \) and \( x_i \in [0,1] \), this indicates that given \( x_i = 1 \) can get the smallest value \( E(x_{i_1}) = 1 - a_{i_1} + \sum_{j=2}^n a_{ij} \leq 1 \).

For any approximate solution \( x = (x_i)_{i \in \mathcal{I}} \) of problem (4), Lemma 3 reveals that there is the smallest value \( E(x_{i_1}) = 1 - a_{i_1} + \sum_{j=2}^n a_{ij} \leq 1 \), if \( a_{i_1} \geq \sum_{j=2}^n a_{ij} \). Moreover, for \( a_{i_1} \leq \sum_{j=2}^n a_{ij} \) or \( a_{i_1} \leq \max_{j=\{2,3,\ldots,n\}} \{a_{ij}\} \), \( E(x_{i_1}) = 1 \) is the smallest value by Lemma 2. Hence, using Lemmas 2 and 3, the following equation can be constructed:

\[
\min \{E(x_{i_1})\} = \min \{1, E(i)\},
\]

(6)

Where \( E(i) = 1 - a_{i_1} + \sum_{j=2}^n a_{ij} \).

**Theorem 2.** For any approximate solution \( x = (x_i)_{i \in \mathcal{I}} \) of problem (4), if \( x^* \) is the best approximate solution of problem (4), then \( D(x^*) = \min D(x) = \min_{i \in \mathcal{I}} \{E(x_i)\} = \min_{i \in \mathcal{I}} \{1, E(i)\} \).

**Proof.** Obviously, combining Eq. (6) with Theorem 1 can obtain this result.

Based on Theorem 2 and Lemma 3, the following results can be easily yielded.

**Lemma 4.** According to the value of \( E(i), \forall i \in \mathcal{I} \), the best approximate solution \( x^* = (x^*_i)_{i \in \mathcal{I}} \) of problem (4) can be determined as one of the following situations:

1. If \( E(i) > 1, \forall i \in \mathcal{I} \), then \( x^* = (x^*_i)_{i \in \mathcal{I}} = 0 \) is the best approximate solution;
2. If \( \min_{i \in \mathcal{I}} \{1, E(i)\} = E(i^*) = 1 \) for some \( i^* \in \mathcal{I} \), then \( x^* = (x^*_i)_{i \in \mathcal{I}} \) contains \( x_{i^*} = 0 \) for all \( k \in \mathcal{I} \) and \( k \neq i^* \) is the best approximate solution;
3. If \( \min_{i \in \mathcal{I}} \{1, E(i)\} = E(i^*) < 1 \) for some \( i^* \in \mathcal{I} \), then \( x^* = (x^*_i)_{i \in \mathcal{I}} \) contains \( x_{i^*} = 1 \) and \( x_k = 0 \) for all \( k \in \mathcal{I} \) and \( k \neq i^* \) is the best approximate solution.
3. Solution algorithm and numerical examples

Based on the above mentioned results, a solution algorithm for problem (4) to solve the best approximate solution of fuzzy relational equations with max-product composition is summarized as follows:

Step 1 : Check the consistency for the provided problem. If it is consistent, then stop.

Step 2 : Compute the values $E(i) = 1 - a_{ii} + \sum_{j=2}^{n} a_{ij}, \forall i \in \mathcal{I}$.

Step 3 : If $E(i) > 1, \forall i \in \mathcal{I}$, then $x^* = (x^*_i)_{i \in \mathcal{I}} = 0$ is the best approximate solution. If $\min_{i \in \mathcal{I}} \{1, E(i)\} = E(i^*) = 1$ for some $i^* \in \mathcal{I}$, then $x^* = (x^*_i)_{i \in \mathcal{I}}$ contains $x^*_{i^*} \in [0, 1]$ and $x_k = 0$ for all $k \in \mathcal{I}$ and $k \neq i^*$ is the best approximate solution. Furthermore, if $\min_{i \in \mathcal{I}} \{1, E(i)\} = E(i^*) < 1$ for some $i^* \in \mathcal{I}$, then $x^* = (x^*_i)_{i \in \mathcal{I}}$ contains $x^*_{i^*} = 1$ and $x_k = 0$ for all $k \in \mathcal{I}$ and $k \neq i^*$ is the best approximate solution.

To get an illustration on how the provided algorithm works, the following numerical examples are presented. First example provides to illustrate that the proposed algorithm can obtain the best approximate solution efficiently. Another example then shows that the problem of fuzzy relational equations with max-product composition contains the empty solution set may have the multiple best approximate solutions.

**Example 1.** Solve the best approximate solution for the following fuzzy relational equations with max-product composition.

$$(x_1 \ x_2 \ x_3) \odot \begin{bmatrix} 0.7 & 0.3 & 0.1 \\ 1.0 & 0.6 & 0.5 \\ 0.8 & 1.0 & 0.2 \end{bmatrix} = (1 \ 0 \ 0)$$

Step 1 : Since this problem is inconsistent, go to the next step.

Step 2 : Compute the values $E(i) = 1 - a_{ii} + \sum_{j=2}^{3} a_{ij}, \forall i \in \mathcal{I}$.

Based on the problem, the values $E(i), \forall i \in \mathcal{I} = \{1, 2, 3\}$ can compute as follows:

$$E(1) = 1 - 0.7 + 0.3 + 0.1 = 0.7, E(2) = 1 - 1.0 + 0.6 + 0.5 = 1.1$$

and $E(3) = 1 - 0.8 + 1.0 + 0.2 = 1.4$.

Step 3 : Since $\min_{i \in \mathcal{I}} \{1, E(i)\} = \min \{1.0, 0.7, 1.1, 1.4\} = 0.7 = E(1) < 1$, by Lemma 4, $x^* = (1, 0, 0)$ is the best approximate solution with the Hamming distance norm $D(x^*) = 0.7$.

**Example 2.** Solve the best approximate solution for the following fuzzy relational equations with max-product composition.

$$(x_1 \ x_2 \ x_3) \odot \begin{bmatrix} 0.9 & 0.7 & 0.2 \\ 1.0 & 0.6 & 0.4 \\ 0.8 & 1.0 & 0.2 \end{bmatrix} = (1 \ 0 \ 0).$$

Step 1 : Since this problem is inconsistent, go to the next step.

Step 2 : Compute the values $E(i) = 1 - a_{ii} + \sum_{j=2}^{3} a_{ij}, \forall i \in \mathcal{I}$.

Based on the problem, the values $E(i), \forall i \in \mathcal{I} = \{1, 2, 3\}$ can compute as follows:

$$E(1) = 1 - 0.9 + 0.7 + 0.2 = 1.0, E(2) = 1 - 1.0 + 0.6 + 0.4 = 1.0$$

and $E(3) = 1 - 0.8 + 1.0 + 0.2 = 1.4$.

Step 3 : Since $\min_{i \in \mathcal{I}} \{1, E(i)\} = \min \{1.0, 1.0, 1.0, 1.4\} = E(1) = E(2) = 1$, by Lemma 4, $x^* = (\alpha, 0, 0), \alpha \in [0, 1]$ and $x^* = (0, \beta, 0), \beta \in [0, 1]$ are the best approximate solutions of this example with the Hamming distance norm $D(x^*) = 1$. 

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4. Conclusions

This study considered the problem of fuzzy relational equations with max-product composition is inconsistent. Some theoretical results show that the value of variable either 1 or 0 is the best approximate solution of the studied problem. Based on these theoretical results, a simple solution algorithm is provided for finding the best approximate solutions. Numerical examples show that the proposed procedure can find out the best approximate solution easily.

Acknowledgment

This research is supported under the Grant of NSC 98-2410-H-238-003.

References