The representation of uncertainty in Operational Research: considerations on the use of possibility, probability, and fuzzy numbers

Matteo Brunelli
IAMSRC and Turku Centre for Computer Science
Åbo Akademi University, Joukainengatan 3-5A, FIN-20520 Åbo, Finland
e-mail: matteo.brunelli@abo.fi

Mario Fedrizzi
Department of Computer and Management Sciences
University of Trento, Via Inama 5, I-38122 Trento, Italy
e-mail: mario.fedrizzi@unitn.it

Abstract

The judgment of the reliability, credibility, or adequacy of the available information plays a critical role when one or more individuals have to make decisions in the presence of uncertainty. In decision making activities, most of the uncertainty comes from subjective judgments and is commonly transmitted through statements in natural language involving vague predicates and therefore linguistic uncertainty is generated, i.e., the uncertainty about a precisely defined quantity that is produced by linguistic information.

In this paper we discuss some issues in the application of possibility and probability theory in the domain of operational research. In doing so, we emphasize how, sometimes, justifications for the use of fuzzy numbers in the representation of uncertainty, lack formality or empirical background.

Keywords: Uncertainty, imprecision, possibility, subjective probability, fuzzy numbers

1. Introduction

Exploring the literature involving the applications of fuzzy sets and possibility theory in Operational Research (OR) the interested reader can easily realize that there is lack of explanations about the motivations supporting the use of fuzzy numbers for representing, most of all when modelling decision making processes, the uncertainty coming from knowledge supplied by the domain expert and based on subjective views. Since information is commonly transmitted through statements in natural language involving vague predicates, linguistic uncertainty is generated, i.e., the uncertainty about a precisely defined quantity that is produced by linguistic information (Walley and de Cooman, 2001). The problem of treating linguistic uncertainty in the framework of fuzzy sets theory was originally addressed by Zadeh (1975) and has further been developed by Zadeh himself (2002).

The main question to answer is if subjective judgements and linguistically expressed information are sufficient conditions to claim the superiority of possibility over subjective probability in the representation of uncertainty. The eminent Bayesian Lindley (1987) claimed

The only satisfactory description of uncertainty is probability...all other methods are inadequate...anything that can be done with fuzzy logic, belief functions, upper and lower probabilities, or any other alternative to probability can better be done with probability.
Even though we cannot accept in toto these kind of sentences, neither can we believe in those who assert that whenever subjective judgements are involved the only effective representation of uncertainty is the one provided by fuzzy numbers. Can we forget Keynes’ lesson (1921) about probability? He claimed that probability should not be identified with statistical frequency. Conversely, it concerns the broader issue of inferring degrees of belief from the available evidence, whereas relative frequencies are a special kind of evidence. Can we miss the seminal contributions of Ramsey (1931), de Finetti (1937) and Savage (1954) who showed that probabilities can be defined in the absence of statistics by relating them to a subjective degree of belief that can be made observable through choice behaviour?

Whatever kind of measure we should adopt for representing uncertainty in constructing our models we cannot forget that the underlying methodological framework should be inspired by the scientific method, and therefore we should go back to the advice of the founding fathers of the field of OR.

In their book, Churchman et al. (1957) defined OR as a “team-based, interdisciplinary application of scientific methods, techniques, and tools to problems involving the operations of a system”. Its main purpose should be to provide managers with a scientific background for carrying out decision processes in order to satisfy some given optimality conditions. As pointed out by Ulrich (2004) “Churchman took the ideal of a scientific approach to managing human affairs seriously enough to follow it through its ultimate consequence.”

It was under the influence of the increasing complexity of economic and social problems and of the systems ideas, developed by Ackoff and Churchman, that OR regained its role of a decision-oriented interdisciplinary activity, embracing the so-called social sciences of economics, psychology, and sociology.

The emerging of problem structuring methods opened the way to ‘soft’ OR approaches, complementary to the classical ‘hard’ OR. Jackson (2006) argued that “these problem structuring methods retained some of the defining characteristics of classical OR, such as rational analysis and modelling, but dropped others which seemed inappropriate given the complexity and uncertainty of social reality”.

To appreciate how experts’ judgments define the decision making path it is crucial to understand what information uncertainty judgments are based on and how this information is used to understand the cognitive substrate of the assessment of uncertainty.

This kind of problem has been addressed for subjective probability judgments and for imprecision and uncertainty as well, and the interested reader may refer to (Clemen and Canan, 2008; Halberg and Teigen, 2009; Nilson et al, 2005; Rakow and Newell, 2010).

In the end, we cannot forget that when handling OR models and decision tools, OR practitioners have an important ethical commitment and they must know that models cannot be value-free, “neither analysis nor modelling work nor the choice of analytical tools is entirely ethically neutral; incomparability, incommensurability and incertitude must be dealt with” (Rauschmayer et al., 2009).

This is the methodological framework that should inspire any approach extending the tools of classical experimental science to the measurement, representation, and management of uncertainty in decision making.

2. Uncertainty and its measures

Using a dictionary, when we examine the various meanings we can easily check that two categories of uncertainty come out (Klir, 1987); vagueness and ambiguity. The concept of vagueness, from the view point of logical analysis, was discussed by Black (1937), who introduced an appropriate symbolism for vagueness by means of which the laws of logic become a point of departure for more elaborate laws able to absorb into a formal system the
deviations from a standard model. According to Klir (1987) ambiguity “is associated with one-to-many relations, i.e., situations with two or more alternatives such that the choice between them is left unspecified”.

If we look at the formal theories of judgment and decisions that have been developed starting from the seminal work of von Neumann and Morgenstern (1944), we realize that all forms of uncertainty have been commonly treated in terms of a single dimension of probability or degree of belief.

Economists, who do not consider what was found by non-economists worthwhile, draw an artificial distinction between Knightian risk (which you can compute) and Knightian uncertainty (which you cannot compute). Knightian uncertainty was named after economist Frank Knight (1885–1972), who, in his book (Knight, 1921), wrote "Uncertainty must be taken in a sense radically distinct from the familiar notion of risk, from which it has never been properly separated...it will appear that a measurable uncertainty, or 'risk' proper, as we shall use the term, is so far different from an unmeasurable one that it is not in effect an uncertainty at all”.

In his book The Black Swan, Taleb (2007b) has developed the so called “black swan theory” where there is no distinction between different kinds of uncertainty. In the section subtitled "The uncertainty of the nerd", he writes “In real life you do not know the odds; you need to discover them, and the sources of uncertainty are not defined.”

The psychological studies of judgment under uncertainty carried out in the second half of the last century by Kahneman and Tversky (Kahneman and Tversky, 1979; 1982; Tversky and Kahneman 1992) provided a comprehensive psychological perspective on uncertainty.

When addressing the problem of mathematical modelling of uncertainty coming from experts’ judgements we have to answer questions like: (i) How can uncertainty be assessed? (ii) What is the most suitable way to measure it? (iii) How can we combine measures of uncertainty? (iv) How can we use it to make inferences and decisions?

The basic issue to address is to define the measures according to some given criteria. Walley (1996) suggested the following six criteria: clarity of interpretation; ability to model judgments expressed in natural language; rules for combining and updating uncertainty; consistency of models and inferences; feasibility of assessment; and feasibility of computations.

According to these criteria one can classify the number of different measures that have been proposed in a wider theory initiated by Dempster (1967) and Shafer (1976) in which, nowadays, probability seems to be a special case of a family of measures.

Apparently, for the last twenty years, since the extensive exposition of Dubois and Prade (1988), possibility theory has proved to be the most valid alternative to probability theory. Having said this, it is important to specify what our observations concern in order to bound the discussion itself. Therefore, before starting the discussion, we need to distinguish between imprecision and uncertainty. As they are both concerning pieces of information, we can use the following approach (Dubois and Prade, 1988). That is, a piece of information, in its simplest formulation, is defined as a quadruple I = {object, attribute, value, confidence}. Imprecision deals with value whereas uncertainty has to do with the confidence. Furthermore, still to bound the discussion, we are aware of the debate which has occurred in literature and of the several clarifications (no need to list them all!), but we prefer to make a step back, as, at a strictly theoretical level, it is nowadays evident that it cannot be stated which theory is superior. Our discussion is then bounded in this sense too: we are not going further than discussing the convenience, and the reasons behind it, of using one approach rather than another.
At this point, it is important to introduce few notions from fuzzy sets theory, not only because it is a tool for dealing with imprecision, but also because it is going to be useful later on, in the kernel of this paper. Given a universe set $X$, a fuzzy set $A$ on $X$ is a set of pairs as follows

$$A = \{(x, \mu_A(x)) | x \in X\}$$

where $\mu_A(x)$ represents the extent to which $x$ belongs to $A$. A function

$$\mu_A : X \rightarrow [0,1]$$

is called a membership function. A fuzzy set is normal if and only if there exists at least one $x$ such that $\mu_A(x) = 1$. The support of a fuzzy set is defined as follows,

$$\text{supp}(A) = \text{cl} \{(x, \mu_A(x) > 0) : x \in X\}$$

where $\text{cl}$ is the topological closure operator.

As far as uncertainty is concerned, let us start with a definition of measure that is general enough to cover our discussion, i.e. a definition where the additivity condition is relaxed. In the following $P(X)$ denotes the power set of $X$.

**Definition 1** (Normalized Monotone Measure). Let $X$ a non-empty set and $C$ any $\sigma$-algebra of its subsets with $X \in C$, then a set function $m : C \rightarrow [0,1]$ is a normalized monotone measure if it satisfies

(i) $m(\emptyset) = 0$, $m(X) = 1$

(ii) $A \subseteq B \Rightarrow m(A) \leq m(B) \quad \forall A, B \in C$

In this framework, a number of measures have been proposed. The most popular one is probability, which is more than three hundred years old. However, the modern definition of probability, due to Kolmogorov, is as it follows.

**Definition 2** (Probability measure). Given two disjoint subsets $A, B \in C$, a probability measure $Pr$ is a normalized monotone measure which satisfies

$$Pr(A \cup B) = Pr(A) + Pr(B) \quad \text{(additivity).}$$

Probability of an event suffices to describe its likelihood. More formally, thanks to additivity property, the dual of any additive measure is the measure itself. Probability is then an autodial measure. In fact,

$$Pr(A) = 1 - Pr(A^c),$$

where $A^c$ is the complement of $A$. Among the other measures, possibility and necessity are likely to be the most popular.

**Definition 3** (Possibility measure). A (normalized) possibility measure, $\text{Pos}$, is a normalized monotone measure defined on $P(X)$ which respects the following property

$$\text{Pos}\left(\bigcup_{i \in I} A_i \right) = \sup_{i \in I} \text{Pos}(A_i)$$

for any family $\{A_i | A_i \in P(X), i \in I\}$, where $I$ is an arbitrary index set.

Unlike probability, possibility is not autodial and it does not suffice to completely describe the likelihood of an event. In fact, possibility is a sub-additive measure as

$$\text{Pos}(A \cup B) \leq \text{Pos}(A) + \text{Pos}(B)$$
holds for all the disjoint sets \( A, B \subseteq X \). A super-additive measure, called necessity, \( \text{Nec} \), for which

\[
\text{Nec}(A \cup B) \geq \text{Nec}(A) + \text{Nec}(B)
\]

holds for all the disjoint sets \( A, B \subseteq X \), is needed in order to complete the description of uncertainty using possibility theory. Necessity and possibility are related according to the following identity

\[
\text{Nec}(A) = 1 - \text{Pos}(\complement A)
\]

Necessity can also be independently defined as

\[
\text{Nec}\left(\bigcap_{i \in I} A_i\right) = \inf_{i \in I} \text{Nec}(A_i)
\]

for any family \( \{A_i | A_i \in P(X), i \in I\} \), where \( I \) is an arbitrary index set.

For a further discussion on normalized measures, the reader can refer to Wang and Klir (2008). Let us just end up recalling that, to each measure, is associated a unique distribution.

3. On the bounded support of L-R fuzzy numbers

Several definitions of fuzzy number have been given in literature, e.g. (Dubois and Prade 1978; 1980; Klir and Yuan 1995) for three different definitions. We disagree on the most restrictive ones, e.g. (Klir and Yuan, 1995), as they require \( \text{supp}(A) \) to be a bounded real interval. This is clearly useful for applications but nevertheless this condition affecting the definition does not seem justified. We believe that, a fuzzy number should be such, if and only if it satisfies the natural semantic of a fuzzy number, e.g. fuzzy subset of all the real numbers close to \( x_0 \in \mathbb{R} \). Hence, semantic, and not computational convenience, should be the only condition to be satisfied to consider a mathematical object to be fuzzy. For this reason we use the following definition of fuzzy number, where the interval \([0, 1]\) can be substituted by any lattice

**Definition 4** ([0,1]-Fuzzy number). A [0,1]-fuzzy number is a convex fuzzy set \( A \) on the real line \( \mathbb{R} \) such that
a) \( \exists \ x_0 \in \mathbb{R}, \mu_A(x_0) = 1 \)
b) \( \mu_A \) is piecewise continuous

If uniqueness in condition a) does not hold, then we rather call it [0,1]-fuzzy real interval. However, hereafter, for sake of simplicity, and without loss of generality, we drop the notation [0,1] and we will simply call them fuzzy numbers and fuzzy real intervals. A special subfamily of fuzzy numbers is that of L-R fuzzy numbers which are defined as follows.

**Definition 5** (L-R fuzzy number). A fuzzy number is called L-R fuzzy number if its membership function is defined as

\[
\mu_A(x) = \begin{cases} 
    L \left( \frac{x - a_1}{a_2 - a_1} \right), & a_1 < x < a_2 \\
    R \left( \frac{a_3 - x}{a_3 - a_2} \right), & a_2 \leq x < a_3 \\
    0, & \text{otherwise}
\end{cases}
\]

with \( a_1, a_2, a_3 \in \mathbb{R}, a_1 < a_2 < a_3 \) and \( L \) and \( R \), strictly decreasing continuous functions from \([0, 1]\) to \([0, 1]\) such that \( L(0) = R(0) = 1 \) and \( L(1) = R(1) = 0 \).
Particularly interesting and widely used in real-world applications, thanks to computational advantages are the so-called triangular fuzzy numbers. That is, a triangular fuzzy number is a L-R fuzzy number with $L$ and $R$ being linear. The above definitions can be extended to the case of fuzzy real intervals. In this case, the definition of L-R fuzzy number is modified accordingly, and if $L$ and $R$ are linear, then the most used name is trapezoidal fuzzy number. Although, in the first place, they are fuzzy sets and, as such, they are representations of imprecision, one can see that their membership function can also be interpreted as a possibility distribution on the real line (Zadeh, 1978).

Our main argument for using definition 4 instead of more restrictive ones is also the reason we question the applicability of L-R fuzzy numbers which, in our opinion are worse in representing “real” possibility than distributions with unbounded support. In the last few years a special emphasis was put on the so called black swans (Taleb 2007). We do not dwell on the concept of black swan and we simply recall that a black swan is an event which is huge in its consequences and a priori unpredictable. Let us start our discussion considering the following indubitable statement

It is not possible that a very unlikely event never happens.

It is straightforward that it implies the following

It must be possible that a very unlikely event happens

To see the direct consequences of these statements, let us think about the application of L-R fuzzy numbers to financial mathematics. Many mathematical models employ L-R fuzzy numbers which, as possibility distributions, entail that

$$x \notin \text{supp}(A) \iff x \notin [a_1, a_2] \iff \text{Pos}(x) = 0.$$  

Usually, when L-R fuzzy numbers are used as possibility distributions the experts are asked for the three real numbers $a_1$, $a_2$, $a_3$. When asked the question, analysts are supposed to give the two least, i.e. $a_1$, $a_3$, and the most possible, i.e. $a_2$, values for a given future cash flow, respectively. The two least possible values represent the boundaries of $\text{supp}(A)$, which means that any event $x \in [-\infty, a_1] \cup [a_3, +\infty]$ must be considered impossible.

L-R fuzzy numbers are simplifications but we wonder whether or not they are too simplistic, especially nowadays because, with the increasing complexity of the world, black swans are occurring more and more often. For this reason, Gaussian shaped probability distributions are claimed to be unreliable and so possibility distributions should be, if we extend our reasoning. However, what would they say about L-R fuzzy numbers whenever they are used as possibility distribution? If risk is naturally underestimated by unbounded distributions, using L-R fuzzy numbers we would underestimate it to an even greater extent!

We do believe that no particular method is going to protect us completely from black swans, but additionally that we should refrain from using techniques which go into the opposite direction. Just as probability distributions, possibility distributions need (fat) tails. A discussion on the importance of tails was offered by Taleb (2007). This need is also stressed by Collan et al. (2009)

We think that the tails should be included in the real option analysis, because even remote possibilities should be taken into consideration.

Our last argument regarding the choice between L-R fuzzy numbers and unbounded distributions is the following. Clearly enough operations on unbounded possibility and probability distributions are more complex. See, for example, mixture operations on probability distributions. The Ockham Razor, *entia non sunt multiplicanda prater necessitatem*, is still a valid postulate but, on the other hand, if two different approaches lead to two different outcomes which can be ordered in terms of reliability, then a major
complexity, as long as it does not become computationally intractable, should be justified. All in all, to cite Zadeh:

Complexity is the price that has to be paid to achieve superior performance.

4. Fuzzy sets and subjective probability

Classical approach to probability, based on frequencies computed over past observations is not the only one. Actually, those scholars claiming the incorrectness of estimating a probability distribution without past observations do not recognize the validity of the subjective approach to probability theory. A grand number of scholars seem to discard the use of probability due to its alleged inadequacy to represent non-stochastic uncertainty. Once again, examples can be drawn from literature without reserve. Although some studies have been carried out to verify the validity of subjective possibility (Raufaste et al. 2003) in a qualitative framework, we disagree on the point of view of those who blindly ignore subjective probability. Our position becomes even stronger if we reason in a quantitative framework, as for instance is a context where fuzzy numbers are used. To this aim, it is the case to recall one of the several debates on fuzzy representations of uncertainty. Originally, Laviolette and Seaman Jr. (1994) made some major comments and one is particularly interesting. Their claim is that scholars in fuzzy sets are mistaken when they state that probability has merely an objective nature. The following statement, from their paper, where FST stands for fuzzy sets theory, is particularly effective.

So where did FST adherents get this notion of probability as exclusively an objective measure of uncertainty? As we are both statisticians, we somewhat reluctantly suggest that they probably (no pun intended) got it from statisticians. Nearly all statisticians and statistical practitioners have been trained primarily in the classical school, which emphasizes the frequency interpretation of probability.

Two among the replies (Dubois and Prade 1994; Klir 1994) are particularly significant. Although Klir (1994) disagreed on most of the points raised by Laviolette and Seaman, he asserted that, as a matter of fact, subjective probability has indeed the same dignity of objective one.

Here, again, I do not subscribe to the alleged position of fuzzy set theorists that probability is exclusively an objective measure of uncertainty. Subjective probability, as exemplified by the Bayesian methodology, is well established and respectable.

In their reply, Dubois and Prade (1994) went even further, stating something which we do not disagree on. Namely,

On the other hand, it is also clear that some fuzzy set advocates are not sufficiently aware of subjective probability.

If possibility has the gift of subjectivity (Dubois et al., 2008), we do not see why probability cannot have it as well. Furthermore, it is also true that probability distributions can be estimated starting from subjective judgements; and several approaches have been proposed so far, e.g. de Finetti’s games (de Finetti, 1931; 1937). Moreover, de Finetti’s approach based on equivalences between gambles seems to be particularly suitable for problems where values on the real line represent amounts of money as, for instance, and once again, financial mathematics. At this point, as it is agreed that probability distributions can, in theory but also
in practice, describe non-stochastic uncertainty, our doubt regards the reason, other than the alleged fact that probability cannot be subjective, leading scholars to use possibility distribution.

The reader interested in finding a way to overcome the contrast between probabilistic and fuzzy methods can look it up in (Coletti and Scozzafava, 2004). In their paper, they proposed an interpretation of fuzzy sets theory in terms of coherent conditional probabilities and their main result is a representation theorem of a possibility function by means of a coherent conditional probability.

5. Imprecise probabilities and the principle of least astonishment

In this last section, before the conclusions, we want to introduce two more points which, according to the fuzzy sets theorists, should contribute to the superiority of possibility over probability. In this very short summary, we claim how, in our opinion, these two issues have not been sufficiently clarified and justified in literature. Let us consider the first (Dubois and Prade, 1988):

From the practical point of view, it is clear that the numbers given by individuals to describe, in terms of probabilities, for example, the state of their knowledge must be considered for what they are, namely, approximate indications. Subjective probability theory does not seem to be concerned with this type of imprecision, considering that a rational individual must be able to furnish precise numbers, when proper procedures for their elicitation are used.

It is easy to (partially) agree on this latter statement. This is the raison d'être of the so called imprecise probabilities, which can overcome this problem. Nevertheless, possibility theory suffers the same drawback, as possibility distributions are also 'precise'. A way to solve this problem is to use type-2 fuzzy sets (Zadeh, 1975). However, in probability theory, second order probabilities (Goodman and Nguyen, 1999) play an almost equivalent role.

Our second point is connected with the principle of least astonishment, also called principle of insufficient reason. Let us cite Dubois and Prade (1988)

First of all, it seems difficult to maintain that every uncertain judgment obeys the rule of betting. The monetary commitment that forms part of the model could prevent an individual from uncovering the true state of his knowledge, for fear of financial loss. Thus, a professional gambler will distribute his stakes evenly if he knows that all the options on which he is betting have equal strength. In the absence of any information, the neophyte will do the same, because it is the most popular strategy. Subjective probabilities allow no distinction between these two states of knowledge and seems ill adapted to situations where the knowledge is sparse.

What has been stated is the well-known Laplace’s principle of least astonishment according to which, if no information is available, then probabilities should be assigned according to a principle of entropy maximization. However, from literature, it is not clear how possibility could yield better results, at least in the case of total ignorance.

From the axioms of normalized measures, it seems that, even if uncertainty was represented by means of possibility theory, the two cases of equal likelihood of the events and total ignorance would be represented in the same way.
6. Conclusions

This paper has focused on the problem of representation and handling of uncertainty generated by subjective judgments expressed by experts when addressing the problem of modeling decision making processes. The paper introduces a framework that places different types of aspects which emphasize the rich and varied role of uncertainty in decision making and can be a useful guide for OR scholars who want to get out of this complex maze. We want to show that some positive statements about the use of possibility theory and fuzzy numbers against subjective probability are ill founded and not inspired by the scientific method that should always enlighten the minds of the operational researchers. Our main points in this paper are that: (i) uncertainty is a rich domain involving imprecision, ambiguity, vagueness, incompleteness coming from knowledge supplied by the domain expert and based on subjective views; (ii) we refuse statements remarking that probability is the only sensible description of uncertainty and is adequate for all problems involving uncertainty; (iii) the superiority of the possibilistic representation of uncertainty with respect to the probabilistic one (subjectivist approach) ought to be refused whenever it is based only on intuition without any support from experimental evidence.

References


