A First Passage Time Problem in System Replacement

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Abstract. A common replacement policy for technical systems is to replace as system by a new one after its economic lifetime, i.e. at that moment, when its long-run maintenance cost rate is minimal. The strict application of the economic lifetime does not take into account the individual, random deviations of maintenance cost rates from the average cost development. To avoid this disadvantage, Beichelt (2001, 2006) suggested a “total maintenance cost limit replacement policy”: A system is replaced as soon as the total maintenance cost spent on it hits a given level. In this contribution, the total maintenance cost development is no longer modeled by functionals of the Brownian motion, but by any stochastic process with non-decreasing sample paths and a convex trend function. Moreover, only the one-dimensional distribution of this process needs to be known. It is proved that with respect to the long-run total maintenance cost rate this policy is superior to the economic lifetime approach. Examples show that applying the total maintenance cost limit replacement policy instead of the economic lifetime leads to cost savings between 4% and 30%. By combining the total maintenance cost limit replacement policy and the well-known “age replacement policy” it is illustrated how the reliability aspect can be included into the model. The simple implementation of the total maintenance cost limit replacement policy, the fact that maintenance cost data are usually available and that no lifetime data are required facilitate its practical application. In: Shebani K (ed), Proceedings of the 1st International Conference on Applied Operational Research – ICAOR (2008), pp 222–229. Lecture Notes in Management Science Vol. 1. ISSN 2008-0050.

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1 Introduction

This contribution focuses on determining cost-optimum replacement times for complex technical systems. One of the first cost-based approaches towards maintenance optimization is due to Lotka (1939): A system is replaced by a new one after its economic lifetime, i.e. at that moment, when its long-run total maintenance cost rate (including replacement cost) is minimal, see also Clapham (1957), Ellon, King, Hutchinson (1966). But the strict application of the economic lifetime does
not take into account the random individual deviations of maintenance cost rates of individual systems from the average cost development. Hence, subsequently the ‘repair cost limit replacement policy’ had been proposed: On a system failure, the corresponding repair cost is estimated. If this cost exceeds a given limit, then the system is replaced by a new one. Otherwise, a repair is carried out. (For recent surveys on this class of maintenance policies and generalizations, see Dohi et al. (1996), Kapur, Garg, Kumar (1999) and Beichelt (2006)). Under this policy, even if the total repair cost justifies a replacement from the cost point of view, a repair may be carried out to bring the system back to work. To overcome the disadvantages of the economic lifetime approach and the repair cost limit replacement policy, Beichelt (2001, 2006) suggested a maintenance policy, which takes into account the whole cost history of the system maintenance:

**Policy 1 (total maintenance cost limit replacement policy):** The system is replaced as soon as the total maintenance cost spent on it reaches or exceeds a given level \( c \). In comparison to the repair cost limit replacement policy, policy 1 has two principal advantages:

1) Applying policy 1 does not require information on the underlying lifetime distribution of the system.

2) Apart from the pure repair costs, costs due continuous monitoring, servicing, stock keeping, personnel cost, loan repayment (including interest) etc. can be taken into account.

From the modeling point of view and with respect to its practical implementation, policy 1 is far more convenient than the repair cost limit replacement policy. It seems to be a suitable basic strategy for planning cost-optimal replacement cycles of complex, wear-subjected technical systems as trucks, cranes, caterpillars, belt conveyors et al. Moreover, taking into account its simple structure and the fact that maintenance cost data are usually available, policy 1 seems to be a suitable basic strategy for planning cost-optimum replacement cycles of whole industrial plants.

Let \( C(t) \) be the total maintenance cost arising in the time interval \([0, t]\), where \( t = 0 \) denotes the time point at which a (new) system starts operating. Then the stochastic process \( \{C(t), \; t \geq 0\} \) describes the cumulative maintenance cost development of the system. For analyzing the policy only the one-dimensional probability distribution of the process \( \{C(t), \; t \geq 0\} \) needs to be known, i.e. the set of distribution functions \( F_t(x) = P(C(t) \leq x), \; t > 0, \; x \geq 0 \). The trend function of \( \{C(t), \; t \geq 0\} \) is denoted as \( M(t) = E(C(t)), \; t \geq 0 \).

**Assumptions**

1) The stochastic process \( \{C(t), \; t \geq 0\} \) has nondecreasing sample paths and starts at the origin: \( C(0) = 0 \).

2) The trend function \( M(t) \) is convex.

3) The planning horizon is infinite.
4) Replacement times are negligibly small.
5) The lengths of replacement cycles (times between two neighbouring replacements) are independent, identically distributed random variables with finite mean.
6) $C(t)$ does not involve replacement costs.

2 Cost Criteria

To determine the long-run maintenance cost per unit time (maintenance cost rate) under policy 1, we need the first passage time $L(c)$ of the stochastic process $\{C(t), t \geq 0\}$ with respect to a positive level $c$. $L(c)$ is defined as $L(c) = \inf\{t, C(t) > c\}$.

Since the sample paths of $\{C(t), t \geq 0\}$ are nondecreasing, the following relationship is obvious: $P(C(t) \leq c) = \{L(c) \geq t\}$ for any $c > 0$ and $t > 0$. Hence, the mean value of $L(c)$ is

$$E(L(c)) = \int_0^\infty G(t)dt.$$  \hspace{1cm} (1)

By the elementary renewal theorem, using assumptions 1 to 4, the long-run total maintenance cost rate under policy 1 is easily seen to be

$$K(c) = \frac{a + c}{E(L(c))}.$$  \hspace{1cm} (2)

where $a_c$, the cost of a replacement, is assumed to be constant. Note that $a + c$ is the total maintenance cost within a replacement cycle and $E(L(c))$ is the mean cycle length. The problem consists in finding a total maintenance cost limit $c = c^*$ which is optimal with respect to $K(c)$. To justify policy 1, it has to be shown that its application is superior to applying the economic lifetime, which requires the same data input. (The author does not know any other replacement policy, which is purely cost based. Efficiency comparisons with other replacement policies, which require information on the lifetime distribution, make little sense.) Hence, in addition to policy 1, the following replacement policy has to be discussed as well:

Policy 2 (economic lifetime): The system is replaced by an equivalent new one after a time span $\tau = \tau^*$ time units, where $\tau^*$ minimizes the criterion

$$K(\tau) = \frac{a \cdot M(\tau)}{\tau}.$$  \hspace{1cm} (3)

Obviously, $K(\tau)$ is the expected long-run maintenance cost rate if the system is always replaced by a new one after $\tau$ time units.

$\tau^*$ is called the economic lifetime of the system. It satisfies equation $dK(\tau)/d\tau = 0$ or if letting $m(\tau) = dM(\tau)/d\tau$, \ldots
\[ \tau m(\tau) - M(\tau) = a. \]  

(4)

In what follows, policy 1 is analyzed for two different models of the one-dimensional probability distribution of the stochastic process \( \{C(t), t \geq 0\} \), and in either case its efficiency is compared to policy 2. Cost comparisons between policy 1 and the repair cost limit replacement policy make little sense since the latter also depends on the underlying lifetime distribution of the system.

3 Cost Comparisons

In this section, the probability distributions of \( C(t) \) are chosen in such a way that simple explicit formulas exist both for the corresponding maintenance cost trend function \( M(t) \) and the mean first passage time \( E(L(c)) \). Less tractable distributions require the use of numerical methods. Of course, the proper stochastic model for describing maintenance cost processes is a compound stochastic process, eventually superimposed by a stochastic process with continuous sample paths. Unfortunately, the first passage time distributions of such processes are usually not known. Instead, rather complicated approximations are offered. For practical applications it is, however, important to point out that this contribution provides the theoretical basis for optimizing policy 1 by simulation.

3.1 Power Distribution

Let, for any positive \( t \), the distribution function of \( C(t) \) be given by

\[ F_t(x) = \begin{cases} \left( \frac{x}{t^y} \right)^2, & 0 \leq x \leq t^y, \quad y > 1, \\ 1, & t^y < x \end{cases} \]

\[ Policy \ 1. \] The probability distribution of the first passage time \( L(c) \) is given by

\[ P(L(c) \geq t) = \begin{cases} \left( \frac{c}{t^y} \right)^2, & c^{1/y} \leq t, \\ 1, & 0 \leq t < c^{1/y} \end{cases} \]

Thus, by (1), \( E(L(c)) = k_1 c^{1/y} \) with \( k_1 = \frac{2y}{2y-1} \). Hence, the corresponding total maintenance cost rate is
\[ K(c) = \frac{1}{k_1} \cdot \frac{a + c}{c^{1/y}}. \] (5)

The optimal limit \( c^* \) and the corresponding minimal total maintenance cost rate are

\[ c^* = \frac{a}{y-1}, \quad K(c^*) = \frac{y}{k_1} \left( \frac{a}{y-1} \right)^{(y-1)/y}. \] (6)

In particular, if \( y = 2 \), then \( c^* = a \). This result is quite intuitive: The total maintenance cost limit should not exceed the cost of a new system.

**Policy 2.** The trend function \( M(t) = E(C(t)) \) is

\[ M(t) = \int_0^t (1 - F_1(x)) \, dx = k_2 \, t^y \]

with \( k_2 = 2/3 \). Hence, when applying the constant replacement interval \( \tau \), the total maintenance cost rate becomes

\[ K(\tau) = \frac{a}{\tau} + k_2 \tau^{y-1}. \]

The corresponding optimal values of \( \tau \) and \( K(\tau) \) are

\[ \tau^* = \left( \frac{a}{k_2(y-1)} \right)^{1/y}, \quad K(\tau^*) = y k_2^{1/y} \left( \frac{a}{y-1} \right)^{(y-1)/y}. \] (7)

**Comparison of policies 1 and 2.** The inequality \( K(c^*) < K(\tau^*) \) is easily seen to be equivalent to

\[ \left( 1 - \frac{1}{2y} \right)^{2y} < \left( \frac{2}{3} \right)^2 \quad \text{or, equivalently,} \quad \left( 1 - \frac{1}{2y} \right)^{y} < \frac{2}{3}. \]

But this inequality is true for all \( y \geq 1 \). In particular, if \( 1 \leq y \leq 5 \), then average cost savings between 14.8 and 2.4% are achieved by applying the optimal cost limit \( c^* \) instead of the economic lifetime \( \tau^* \).

### 3.2 Frechet Distribution

Let, for any positive \( t \), distribution function and density of \( C(t) \) be given by

\[ F_t(x) = \exp \left( - \left( \frac{t^y}{x} \right)^2 \right) \quad \text{and} \quad f_t(x) = 2t^2 x^{-3} \exp \left( - \left( \frac{t^y}{x} \right)^2 \right); \quad x > 0, \ y > 1. \]
Policy 1. By (1), the mean first passage time is seen to be

$$E(L(c)) = k_1 c^{1/y} \quad \text{with} \quad k_1 = \Gamma \left( \frac{1}{2y} + 1 \right).$$  \hspace{1cm} (8)

Thus, the corresponding maintenance cost rate $K(c)$ has again structure (5) so that the optimal values of $c$ and $K(c)$ are given by (6) with $k_1$ given by (8).

Policy 2. The trend function $M(t) = E(C(t))$ is $M(t) = E(C(t)) = k_2 t^y$ with $k_2 = \sqrt{\pi}$ so that $K(t) = \frac{a}{t} + k_2 t^{y-1}$.

Thus, apart from the value of the constant $k_2$, the optimal values of $\tau$ and $K(\tau)$ are again given by (7).

Comparison of policies 1 and 2. For all $y > 1$, the relationship $K(c^*) < K(\tau^*)$ is equivalent to

$$1 < \Gamma(1.5) \left( 1 - \frac{1}{2y} \right)^{1/y} \quad \text{or} \quad 2 / \sqrt{\pi} < B(x) = \left[ \frac{\Gamma(x)}{\Gamma(1-x)} \right]^{1/2(1-x)}, \quad 0.5 \leq x < 1.$$  \hspace{1cm} (9)

Note that $B(x)$ is a decreasing function in $[0.5, 1)$ with

$$\lim_{t \to \infty} B(x) = e^{E/2} > 2 / \sqrt{\pi},$$

where $E \approx 0.5772$ is the Euler number. Hence, (9) holds for all $y > 1$ so that policy 1 is superior to policy 2. In particular, if $1.1 \leq y \leq 5$, then average cost savings between 4.2% and 31% are achieved by applying the optimal cost limit $c^*$ instead of the economic lifetime $\tau^*$.

4 Combined Age-Total Repair Cost Limit Replacement Policy

Scheduling replacements based on a total maintenance cost limit does not take into account, at least not explicitly, reliability requirements imposed on the system. There are several possibilities for including the reliability aspect into the model. A simple way consists in limiting the length of a replacement cycle by a constant $\tau$. By a suitable choice of $\tau$, severe breakdowns of the system can be avoided with high probability. However, a 'suitable choice' of $\tau$ requires some prior knowledge on the lifetime distribution of the system.

Policy 3. The system is replaced as soon as the total maintenance cost reaches or exceeds a level $c$ or after $\tau$ time units, whichever occurs first.
Notwithstanding a formal analogy, this policy strongly deviates from the common, lifetime-based ‘age replacement policy’. Following policy 3, the cycle length \( Y = \min \{ L(c), \tau \} \) (time between two neighboring replacements) has distribution function and mean value

\[
P(Y \leq t) = \begin{cases} 
1 - F_t(c) & \text{for } 0 \leq t \leq \tau \\
1 & \text{for } t > \tau 
\end{cases}, \quad E(Y) = \int_0^\tau F_t(c) \, dt.
\]

In view of the assumptions stated in section 1, the long-run maintenance cost rate has structure

\[
K(c, \tau) = \frac{(a + E[C(\tau) \mid L(c) > \tau])P(L(c) > \tau) + (a + c)P(L(c) \leq \tau)}{\int_0^\tau F_t(c) \, dt}.
\]

Since

\[
E[C(\tau) \mid L(c) > \tau] = E[C(\tau) | C(\tau) < c] = \frac{1}{F_t(c)} \int_0^c F_t(x) \, dx,
\]

the long-run maintenance cost rate becomes after some simple algebra

\[
K(c, \tau) = \frac{a + c F_{\tau}(c) + \int_0^c F_t(x) \, dx}{\int_0^\tau F_t(c) \, dt}.
\]

Under the assumptions in section 3.1, explicit formulas for the optimum vector \((c, \tau) = (c^*, \tau^*)\) can be obtained.

5 Conclusions

The replacement policy proposed is based on limiting the cumulative maintenance cost within a replacement cycle. Its simple structure, the fact that its application does not require information on the underlying lifetime distribution of the system and that maintenance cost data are usually available, facilitates its practical application for scheduling replacements of complex, wear-subjected technical systems and for determining cost-optimal overhaul time points of whole industrial plants. The examples analyzed provide strong arguments in favor of scheduling replacements on the basis of cumulative maintenance cost limits instead of the economic lifetime. However, more work, in particular Monte-Carlo simulation, needs to be done to get deeper insight into the relationship between the policy proposed and the economic lifetime approach.
References


