An Optimal Common Set of Weights for Integrated Voting Model

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Abstract. This paper shows the problem of finding the most desirable candidate in a voting system can be integrated into a minimax linear programming model. The paper proposes an optimal common set of weights for the integrated minimax voting LP model. Further it is shown that the integrated model is feasible and capable to rank the most desirable candidate(s).

Keywords: Integrated voting model, most desirable candidate, linear programming.

1 Introduction

In voting systems, one candidate may receive different votes in different ranking places. The total score of each candidate is the weighted sum of the votes he/she receives in different places. The winner is the one with the biggest total score. So, the key issue is how to determine the weights associated with different places. As an application of data envelopment analysis (DEA) [1], Cook and Kress [2] originated a general model for aggregating votes from a preferential ballot. Their model is presented which aggregates votes into overall index in a way that allows each candidate to be assessed [2]. Following the seminal paper of Cook and Kress [2] a number of research papers have been listed in the literature of voting model. Green et al. [3] used the so-called cross-evaluation matrix as the summary of a self- and peer-rating process in which the candidates seek to interpret the voters preferences as favorably for themselves, relative to the other candidates, as possible. Their paper also applied to the selection of R & D projects to comprise an R & D program, thus indicating their wider applicability [3]. Noguchi et al. [4] utilized the cross-efficiency evaluation technique in voting. To identify the voting winner Hashimoto [5] proposed a super-efficiency model. Obata and Ishii [6] suggested
the normalized weights to discriminate the DEA efficient candidates. Liu and Hai [7] introduced the voting analytic hierarchy process (AHP) method for selecting supplier. For a ranked voting system Foroughi and Tamiz [8] proposed an effective total ranking model. Recently Wang and Chin [9] acquainted the discriminating DEA efficient candidates using their least relative total scores. In order to determine the most desirable candidate(s) the exiting voting models need to solve \( m \) linear programming (LP), one LP for each candidate. The aim of this paper is introducing an integrated minimax voting model for finding the most desirable candidate by solving only a single LP. Thus, the main contribution of this work is that it finds a common set of optimal weights which evaluates all the candidates in a fair manner. The paper also reduces the computational complexity of solving the preference voting models as well. Two numerical examples illustrate the application of the minimax model and also the verification of the model’s validity in generating results. The remaining of the paper is organized as follows:

Section 2 gives a brief explanation of the original voting model. Section 3 introduces an integrated minimax voting LP model and its capabilities. A new voting ranking approach is given in Section 4. The model is illustrated on published data in Section 5. Conclusion is given in the final section.

## 2 Standard voting model

In the preference voting framework each candidate \( i = 1, \ldots, m \) receives \( v_{ij} \) of the \( j^{th} \) place votes for \( j = 1, \ldots, k \). To evaluate the relative efficiency of each candidate using DEA it is assumed candidate \( i_0 \) denotes \( DMU_{i_0} \). In addition \( v_{ij} \) is the \( j^{th} \) output of \( DMU_{i_0} \) and all the DMUs have the same single input, arbitrary one. So the voting data can be showed as the following DEA observations:

<table>
<thead>
<tr>
<th>DMU</th>
<th>Output(_1)</th>
<th>...</th>
<th>Output(_k)</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td>( DMU_i )</td>
<td>( v_{i1} )</td>
<td>...</td>
<td>( v_{ik} )</td>
<td>1</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>...</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( DMU_{i_0} )</td>
<td>( v_{i_01} )</td>
<td>...</td>
<td>( v_{i_0k} )</td>
<td>1</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>...</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( DMU_m )</td>
<td>( v_{m1} )</td>
<td>...</td>
<td>( v_{mk} )</td>
<td>1</td>
</tr>
</tbody>
</table>

Therefore the standard DEA model for \( DMU_{i_0} \) can be written as:
\[
\begin{align*}
\text{max} \quad & \sum_{j=1}^{k} v_{i,j} w_j \\
\text{s.t.} \quad & v = 1 \\
& \sum_{j=1}^{k} v_{i,j} w_j - v \leq 0 \quad i = 1, \ldots, m \\
& w_j \geq 0 \quad j = 1, \ldots, k \\
& v \geq 0
\end{align*}
\]

Imposing the weight restrictions [10-11] to model (1) gives the following voting model.

\[
\begin{align*}
\text{max} \quad & \sum_{j=1}^{k} v_{i,j} w_j \\
\text{s.t.} \quad & \sum_{j=1}^{k} v_{i,j} w_j \leq 1 \quad i = 1, \ldots, m \\
& w_j - w_{j+1} \geq d(j, \varepsilon) \quad j = 1, \ldots, k - 1 \\
& w_k \geq d(k, \varepsilon)
\end{align*}
\]

Where \(d(j, \varepsilon)\) is discrimination intensity function and \(\varepsilon\) is discriminating factor. This model is the original voting model proposed by Cook and Kress [2]. In model (2) for any given set of weights (multipliers) \(w_j\) they [2] defined the desirability index for candidate \(i_0\) as

\[
z_{i_0} = \sum_{j=1}^{k} v_{i_0,j} w_j
\]

The objective function of the model seeks to determine a set of multipliers which maximizes the desirability of candidate \(i_0\). Also the weight restrictions added to the model for each candidate is permitted to choose the most favourable weights to be applied to his/her standings (first place, second place, etc.) with the restrictions that the weight for a \(j\) place vote should be more than that for a \(j + 1\) amount.
3 Integrated minimax voting model

We define a new variable, \( d_{i_0} \), the desirability deviation of candidate \( i_0 \) as
\[
d_{i_0} = 1 - z_{i_0}.
\]
Therefore, maximizing the desirability index of candidate \( i_0 \) is equivalent to minimizing \( d_{i_0} \). That is
\[
\min d_{i_0} \\
\text{s.t.} \\
\sum_{j=1}^{k} v_{ij} w_j + d_i = 1 \quad i = 1, \ldots, m \\
w_j - w_{j+1} \geq d(j, \varepsilon) \quad j = 1, \ldots, k-1 \\
w_k \geq d(k, \varepsilon) \\
d_i \geq 0 \quad i = 1, \ldots, m
\]  

(3)

The objective functions of formulations (2) and (3) are specific to a particular candidate. Therefore for determining all of the desirable candidates we need to solve LP model (3) \( m \) times, each aiming to minimize the deviation from desirability for a particular candidate. Minimax desirability measure can be briefly defined as the minimization of the maximum deviation from desirability among all candidates. Further discrimination among candidates can be allowed by replacing the objective function of formulation (3) with the minimax desirability measure, which yields the following minimax desirability model.

\[
\min D \\
\text{s.t.} \\
D - d_i \geq 0 \quad i = 1, \ldots, m \\
\sum_{j=1}^{k} v_{ij} w_j + d_i = 1 \quad i = 1, \ldots, m \\
w_j - w_{j+1} \geq d(j, \varepsilon) \quad j = 1, \ldots, k-1 \\
w_k \geq d(k, \varepsilon) \\
d_i \geq 0 \quad i = 1, \ldots, m
\]

(4)
Where \( D = \max \{ d_i : i = 1, \ldots, m \} \). Let \( x^* = (D^*, d^*, w^*) \) denotes an optimal solution of model (4). Define the corresponding set \( Q(x^*) = \{ i : d_i^* = 0 \} \). The following lemma proves that at least one of the optimal deviations is zero.

**Lemma 1:** For any optimal solution of model (4), say \( x^* \), we have \( |Q(x^*)| \geq 1 \).

**Proof:** Consider the dual of (4) shown below:

\[
\max \sum_{i=1}^{m} \beta_i + \sum_{j=1}^{k} d(j, \varepsilon) \gamma_j
\]

s.t.
\[
\sum_{j=1}^{m} \alpha_j = 1
\]
\[
\beta_i - \alpha_j \leq 0 \quad i = 1, \ldots, m
\]
\[
\sum_{i=1}^{m} v_{ij} \beta_i + \gamma_j \leq 0
\]
\[
\sum_{i=1}^{m} v_{ij} \beta_i - \gamma_{j-1} + \gamma_j \leq 0 \quad j = 2, \ldots, k
\]
\[
\alpha_j \geq 0 \quad i = 1, \ldots, m
\]
\[
\gamma_j \geq 0 \quad j = 1, \ldots, k
\]

On the contrary assume \( |Q(x^*)| = 0 \). It implies that \( d_i^* > 0 \) for each \( i = 1, \ldots, m \). According to the complementary slackness conditions the corresponding dual constraints must be binding at optimality. That is

\[
\beta_i^* - \alpha_j^* = 0 \quad \text{for all } i
\]

Therefore the following LP model should be feasible.
\[ \max \sum_{i=1}^{m} \beta_i + \sum_{j=1}^{k} d(j, \varepsilon) \gamma_j \]

s.t.,

\[ \sum_{i=1}^{m} \alpha_i = 1 \]
\[ \beta_i - \alpha_i = 0 \quad i = 1, \ldots, m \]
\[ \sum_{i=1}^{m} v_{i1} \beta_i + \gamma_1 \leq 0 \]
\[ \sum_{i=1}^{m} v_{ij} \beta_i - \gamma_{j-1} + \gamma_j \leq 0 \quad j = 2, \ldots, k \]
\[ \alpha_i \geq 0 \quad i = 1, \ldots, m \]
\[ \gamma_j \geq 0 \quad j = 1, \ldots, k \]

Note that the only difference between models (5) and (5') is the second constraint of the models. Accordingly the dual of the above model is model (4') except the deviation variables are free in sign for model (5').

Denote the dual of model (5') by (4'). Decreasing \( d_i \rightarrow -\infty \) and increasing \( w_j \rightarrow +\infty \) simultaneously makes (4') to an unbounded problem. Therefore the corresponding dual model should be infeasible one which is a contradiction.

**Lemma 2:** At least one of the candidates evaluated by model (4) is desirable.

**Proof:** Using Lemma 1 the proof is straightforward.

### 4 New ranking approach

This section introduces a new integrated voting measure. By solving only a single model, it provides a common set of optimal weights to find the most desirable candidate(s). Suppose for a given optimal solution \( \mathbf{x}^* = (D^*, d^*, w^*) \) of model (4) \( Q(\mathbf{x}^*) \geq 2 \). The objective is to find the most desirable candidate(s). We propose the following model.
\[
\begin{align*}
\min D \\
\text{s.t.} \\
D - d_i & \geq 0 \quad i = 1, \ldots, m \\
\sum_{j=1}^{k} v_{ij}w_j + d_i & = 1 \quad i = 1, \ldots, m \\
\sum_{i=1}^{m} \delta_i & = m - 1 \\
\delta_i - d_i \lambda_i & = 0 \quad i = 1, \ldots, m \\
w_j - w_{j+1} & \geq d(j, \varepsilon) \quad j = 1, \ldots, k - 1 \\
w_k & \geq d(k, \varepsilon) \\
\delta_i & \in \{0,1\} \quad i = 1, \ldots, m \\
d_i & \geq 0, \lambda_i \geq 1 \quad i = 1, \ldots, m
\end{align*}
\]

Where the nonlinear constraints \( \delta_i - d_i \lambda_i = 0 \) (for each \( i \)) and summation of the binary variables to \( m-1 \), \( \sum_{i=1}^{m} \delta_i = m - 1 \), assure only one of the deviation variables is allowed to vanish. Now consider the following theorem.

**Theorem:** Model (6) is feasible.

**Proof:** Using Lemma 1 and 2 the proof is clear. ■

## 5 Numerical illustration

To illustrate the procedure of finding the most desirable candidate(s) shown at the paper, consider the data used by Cook and Kress [2], containing four candidates. Let the 1st and 2nd place standings be given as shown in table 2.

<table>
<thead>
<tr>
<th>Candidate</th>
<th>( v_{11} )</th>
<th>( v_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>
To detect the most desirable candidate, model (4) becomes as

\[
\begin{align*}
\min & \quad D \\
\text{s.t.} & \quad D - d_1 \geq 0, \quad D - d_2 \geq 0 \\
& \quad D - d_3 \geq 0, \quad D - d_4 \geq 0 \\
& \quad 6w_1 + 8w_2 + d_1 = 1, \quad 4w_1 + 11w_2 + d_2 = 1 \\
& \quad 8w_1 + 2w_2 + d_3 = 1, \quad 3w_1 + d_4 = 1 \\
& \quad w_1 - w_2 \geq \varepsilon, \quad w_2 \geq \varepsilon \\
& \quad d_i \geq 0 \quad i = 1, 2, 3, 4
\end{align*}
\]

For convenience we use \( d(j, \varepsilon) = \varepsilon \) instead of the discrimination intensity function. In a voting application an appropriate discriminating function according to the opinions of decision maker(s) should be selected. It usually depends on the behaviors of the underlying data set. The choice of form for \( d(., \varepsilon) \) and the value of \( \varepsilon \) are two existing issues in the voting models. For the above model the maximum discriminating factor in the DEA models is proposed by Amin and Emrouznejad [12] as the following compact mathematical form

\[
\varepsilon^* = \min \left\{ \left( \sum_{j=1}^{k} (k - j + 1)w_{ij} \right)^{-1} : \ i = 1, 2, 3, 4 \right\} = 0.05
\]

Considering \( \varepsilon^* = 0.05 \) in the model, gives the below optimal solution shown in table 2:

<table>
<thead>
<tr>
<th>Table 3. The optimal solution.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 3 indicates that the most desirable candidate is candidate 1. The following table is extracted from Obata and Ishii [6].
Table 3. Seven candidates with two places.

<table>
<thead>
<tr>
<th>Candidate</th>
<th>First Place</th>
<th>Second Place</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>32</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>28</td>
<td>20</td>
</tr>
<tr>
<td>C</td>
<td>13</td>
<td>36</td>
</tr>
<tr>
<td>D</td>
<td>20</td>
<td>27</td>
</tr>
<tr>
<td>E</td>
<td>27</td>
<td>19</td>
</tr>
<tr>
<td>F</td>
<td>30</td>
<td>8</td>
</tr>
<tr>
<td>G</td>
<td>0</td>
<td>30</td>
</tr>
</tbody>
</table>

As the previous example, let \( d(j, \varepsilon) = \varepsilon^* = 0.01 \). An optimal solution of the corresponding model (4) becomes in Table 4.

Table 4. An optimal solution.

<table>
<thead>
<tr>
<th>( w_1^* )</th>
<th>( w_2^* )</th>
<th>( d_A^* )</th>
<th>( d_B^* )</th>
<th>( d_C^* )</th>
<th>( d_D^* )</th>
<th>( d_E^* )</th>
<th>( d_F^* )</th>
<th>( d_G^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0258</td>
<td>0.0138</td>
<td>0.0350</td>
<td>0</td>
<td>0.1662</td>
<td>0.1098</td>
<td>0.0397</td>
<td>0.1143</td>
<td>0.5850</td>
</tr>
</tbody>
</table>

So, the desirable candidate is candidate B as shown in the optimal solution. Generally, for a given optimal solution \( \mathbf{x}^* \), if \( |Q(\mathbf{x}^*)| \geq 2 \) one can use model (5) to rank the desirable candidates.

6 Concluding Remarks

This paper started with the motivation for the standard voting models, reformulated it as an integrated minimax problem. This model was demonstrated to be analytically feasible. The proposed model extracted the most desirable candidate(s) only by solving one model. Thus, the main contribution of this work is that it reduces the computational complexity performance of the preference voting. It has illustrated the application of the model by two numerical examples and verified its validity in generating valid most desirable candidate(s).
References