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An $O(T^3)$ algorithm for the capacitated economic lot-sizing problem with stationary capacities and concave cost functions with non-speculative motives

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Abstract. We consider the capacitated economic lot-sizing problem (CLSP) with stationary capacities and concave cost functions with non-speculative motives. Under these assumptions we show that there is an optimal solution of the problem that is composed only by subplans that can be computed in linear time, which means that the problem can be solved in $O(T^3)$ computation time.

Keywords: capacitated economic lot-sizing problem; inventory control; optimization

Introduction

The capacitated economic lot-sizing problem (CLSP) refers to the problem of determining the quantities to produce at each period in order to meet the demand requirements of a single product on time, minimizing the sum of the costs involved. The numbers of units that can be produced at each period are limited by a maximum value. The CLSP is an NP-hard problem in general, and even for special cases on the cost functions and/or the capacity pattern (Bitran and Yanasse, 1982; Florian et al. 1980). For the case of concave cost functions and stationary capacities (i.e., equal capacity upper-bounds for each period) Florian and Klein (1971) develop an effective algorithm of $O(T^4)$ time. More recently, faster algorithms of $O(T^3)$ and $O(T^2 \log T)$ times have been suggested by van Hoesel and Wagelmans (1996) for the case of linear inventory holding costs and by van Vyve (2007) for the case of linear costs with non-speculative motives, respectively. Bitran and Yanasse (1982) propose several polynomial time algorithms for different cases of set-up, holding and unit production costs, and different capacity patterns. Chung and Lin (1988) provide an algorithm of $O(T^2)$ time for the CLSP with non-increasing set-up and unit production costs and non-decreasing capacity pattern. van den Heuvel and Wagelmans (2006) also consider this problem, providing other $O(T^2)$ time algorithm which may run faster in practice. Chen et al. (2008) provide a pseudo-polynomial time algorithm for the same CLSP case but with more general cost functions. More recently, Ng et al. (2010) suggest an approximation algorithm for the CLSP with backlogging and a monotone cost structure and Okhrin and Richter (2011) analyze the CLSP with minimum order quantity restrictions. For surveys on the CLSP and extensions, we refer the readers to Brahimi et al. (2006), Karimi et al. (2003), Pan et al. (2009) and Robinson et al. (2009).

The main contribution of this paper is to show that the subplans composing an optimal solution of the CLSP with stationary capacities and concave cost functions with non-speculative motives (i.e. when it is profitable to produce as late as possible) have a particular structure and can be obtained by means of a linear time procedure. This result implies that the running time of the well-known algorithm of Florian and Klein (1971) for the CLSP can be improved from $O(T^4)$ time to $O(T^3)$ time for the case of non-speculative motives on the costs. According to our best knowledge, our approach can be applied over situations that are not covered by previous related works in the literature. In addition, we want to note that our approach is simpler than the approach of Van Vyve (2007).

The remainder of the paper is organized as follows. Section 2 provides the notation and the mathematical formulation for the problem and some basic properties of the CLSP. In Section 3 we prove that there is an optimal solution of the CLSP with stationary capacities and concave cost functions with non-speculative motives, which is composed only by a particular kind of subplans. In Section 4 we describe the linear time procedure for obtaining this particular kind of production subplans which means that the problem can be solved in $O(T^3)$ time. Finally, Section 5 concludes the paper.

Notation and mathematical formulation

We consider the CLSP with a finite planning horizon length $T > 0$. For each period $t = 1,\ldots,T$, there is a known demand requirement $D_t \geq 0$ which must be satisfied on time by producing on the same period or in a previous one the quantity $x_t \geq 0$. Backlogging demand is not allowed and the production quantity at each period is limited by C_t with $0 < C_t < +\infty$ and $t = 1,\ldots,T$. There are costs for carrying out the production and for storing a positive quantity $y_t \geq 0$ at each period $t = 1,\ldots,T$. Henceforth, we assume that the production cost function $f_t(\cdot)$ and the holding inventory cost function $h_t(\cdot)$ are non-decreasing concave functions on the interval $[0,+\infty)$, and equal to zero when its argument is zero or negative, with $t = 0,\ldots,T$. It is also assumed that the initial inventory and the lead-time are equal to zero. The objective is to determine the quantities x_t to produce at each period in order to meet the demand requirements on time fulfilling the capacity constraints and minimizing all the involved costs. The problem described above can be formulated as the following Mixed Integer Linear Programming (MILP) problem:

$$\min \sum_{t=1}^{T} \{ f_t(x_t) + h_t(y_t) \} \tag{P}$$

subject to:

$$y_t = y_{t-1} + x_t - D_t, \quad \forall t = 1,\ldots,T \tag{1}$$

$$y_0 = 0 \tag{2}$$

$$x_t \le C_t, \quad \forall t = 1,\ldots,T \tag{3}$$

$$x_t, y_t \ge 0, \quad \forall t = 1,\ldots,T \tag{4}$$

Constraint (1) is the inventory equilibrium equation that ensures that demand is met. Constraint (2) states that the initial inventory quantity must be zero. Constraint (3) represents the production capacity limit and constraint (4) the set of possible values for the decision variables. Since the decision variables y_t can be replaced by $(x_{1t} - D_{1t})$, with x_{ij} and D_{ij} the accumulated production and demand between periods *i* and *j* respectively, with $1 \le i \le j \le T$, the problem formulated above reduces to find the set of feasible plans $x = (x_1,\ldots,x_T)$. The set of feasible plans is not empty if and only if the accumulated demand of the first *t* periods does not exceed the accumulated capacities over these periods, formally:

$$\sum_{i=1}^{t} C_i \ge \sum_{i=1}^{t} D_i, \quad \forall t = 1,\ldots,T \tag{5}$$

Henceforth we assume that expression (5) is fulfilled. Since the objective function of (P) is a concave function and the constraints (1) – (4) define a closed bounded convex set, there is an optimal solution of the CLSP that is an extreme point of this set. Florian and Klein (1971) proved that the extreme-point solutions are composed only by subplans $S_{ii} = (x_{i+1},\ldots,x_j)$ called *capacity constrained sequence* such that $y_i = y_j = 0$ and $y_t > 0$, for all *t* in $0 \le i < t < j \le T$, and the production quantities of the periods are zero or equal to the capacity, except in at most one period. Based on this result, they provide an $O(T^4)$ time algorithm for solving the CLSP with stationary capacities and general concave cost functions. Thus, for the remainder of the paper we consider a stationary capacity-pattern, i.e., $C_t = C$ for all $t = 1,\ldots,T$.

In the following sections, we show that the algorithm of Florian and Klein (1971) can be improved from $O(T^4)$ to $O(T^3)$ time if it is also assumed a non-speculative type of cost structure.

The ascending capacity constrained sequences of the CLSP

In this section we prove that for the CLSP with non-speculative motives on the costs and stationary capacity-pattern, there is an optimal solution that is composed only by particular subplans that we refer as ascending capacity constrained sequences, since the production quantities in this kind of sequences is increasing over time. This kind of sequences was introduced in Chung and Lin (1988). We begin providing the definitions needed for the proof.

Definition 1. We say that the cost functions of the CLSP are *non-speculative* if the expressions below are fulfilled:

$$f_i(a) + \sum_{t=i}^{j-1} h_t(b_t) \geq \sum_{t=i}^{j-1} h_t(b_t - a) + f_j(a), \tag{6.1}$$

$$\text{with } b_t \geq a > 0 \text{ integers}, 1 \leq i \leq t < j \leq T$$

$$f_i(a) + \sum_{t=i}^{j-1} h_t(b_t) + f_j(c) \geq f_i(a-1) + \sum_{t=i}^{j-1} h_t(b_t - 1) + f_j(c+1), \tag{6.2}$$

$$\text{with } a, b_t, c > 0 \text{ integers}, 1 \leq i \leq t < j \leq T$$

Expression (6.1) states that it is profitable to transfer forward all the production quantity from one active period to another period initially inactive, and (6.2) that it is profitable to transfer forward at least one unit between two active periods. Expressions (6.1) and (6.2) are fulfilled in different settings of interest, e.g., when all the costs involved are concave functions and either stationary or non-increasing. In particular, we note that there are cost structures satisfying Definition 1 which are not covered by previous related works. For instance, consider a CLSP instance of T periods where $f_t(x) = K_t^P + c_t^P x$, with $K_t^P = t$, $c_t^P = 1$, the set-up and unit costs for production respectively, $h_t(x) = 2x + \sqrt{x}$, and stationary capacity C, i.e., $x_t < C$, for each period $t = 1,\ldots,T$ respectively. This particular cost structure satisfies expressions (6.1) and (6.2) but it is not supported by the algorithms of Chung and Lin (1988), van den Heuvel and Wagelmans (2006), Chen et al. (2008), van Hoesel and Wagelmans (1996) and Van Vyve (2007).

Definition 2. We say that a capacity constrained sequence $A_{ij} = (x_{i+1},\ldots,x_j)$ is an *ascending capacity constrained sequence* (ACC sequence) whenever the period with a positive quantity below capacity, if it exists, is the first among all the positive periods in the sequence, i.e. $x_{i+1} \leq \ldots \leq x_{j-1} \leq x_j$, $0 \leq i < j \leq T$.

Proposition 1. Consider the CLSP with stationary capacities and non-speculative motives on the costs according to Definition 1. Then, the solutions of the CLSP composed only by ACC sequences according to Definition 2 are dominant, i.e., given a feasible solution of the CLSP for which there is at least one sequence that is not an ACC sequences, we can determine a new feasible solution composed only by ACC sequences with at most the same cost than the original.

Proof. Consider a feasible solution $x = (x_1,\ldots,x_T)$ of the CLSP composed only by capacity constrained sequences. Without loss of generality, suppose that *x* has only one sequence $S_{\alpha\beta}$ that is not an ACC sequence (the fractional period is not the first among the positive periods in the sequence). This means that there are two consecutive periods *i* and *j* such that $C = x_i > x_j > 0$ with $0 \le \alpha \le i < j \le \beta \le T$. Then, define $\varepsilon = \min\{y_i, y_{i+1},\ldots,y_{j-1}, C - x_j\}$ and consider the following definition of a new feasible solution z :

$$\begin{cases} z_i = x_i - \varepsilon \\ z_j = x_j + \varepsilon \\ z_t = x_t, \quad t \ne i, t \ne j, 1 \le t \le T \end{cases}$$

We note that one of the two following cases is fulfilled for *z*: 1) $z_j = C$; or 2) $y_t = 0$, for some *t* in $i \le t < j$. If case 1) is fulfilled, then the production quantity of period *i* in the new solution z is below capacity since $0 < \varepsilon < C$. In the case that period *i* is not the first positive period in the sequence, we can determine a new ε for period *i* and the immediately previous period *k* of the sequence such that $C = x_k > x_i > 0$, with $\alpha \le k < i \le \beta$. We repeat this process until the first positive period in the sequence is reached. On the other hand, if case 2) is fulfilled, we note that the sequence $S_{\alpha\beta}$ has been decomposed into two new sequences $S_{\alpha t}$ and $S_{t\beta}$ for some *t* with $i \le t < j$. We note that sequence $S_{t\beta}$ is an ACC sequence, since all the positive periods are at capacity. In the case of the sequence $S_{\alpha t}$, the period *i* is below capacity. If it is not the first positive period we proceed as we explained for case 1) for period *i* and the immediately previous period *k* of the sequence for which $C = x_k > x_i > 0$. Since we are assuming that the costs are non-speculative according to Definition 1, the cost of the new solution *z* is at most equal to the cost of the original solution x. Thereby, we have constructed another feasible solution with at most the same cost as the original one but composed only by ACC sequences. ■

The values of an ascending capacity constrained sequence

In this section we describe a procedure for determining the values of an ACC sequence in linear time. First, by Florian and Klein (1971), we note that for any capacity constrained sequence $S_{ij} = (x_{i+1},\ldots,x_j)$, there are *K* periods at capacity, at most one positive period below capacity and the remaining periods equal to zero, with $x_{i+1} + \ldots + x_j = D_{ij} = K.C + \varepsilon$, with $K \in \{1,2,\ldots\}$ and $\varepsilon \geq 0$. Then, in order to compute the values of an ACC sequence between any pair of periods *i* and *j*, we must determine a sequence $A_{ij} = (x_{i+1},\ldots,x_j)$ satisfying 1) $x_{(i+1)j} = D_{(i+1)j} = K.C + \varepsilon$, with $K \in \{1,2,\ldots\}$, $\varepsilon \geq 0$; 2) $y_t > 0$ with $i \leq t < j$; and 3) $x_{i+1} \leq \ldots \leq x_{j-1} \leq x_j$. Without loss of generality assume that $D_{i+1} > 0$. If $\varepsilon > 0$, then $x_{i+1} = \varepsilon$, otherwise $x_{i+1} = C$. The next positive period *t* at capacity, i.e., $x_t = C$, will be the earliest period *t* such that $D_{(i+1)t} > x_{i+1}$, with $i < t \leq j$. We apply the same reasoning until all the *K* positive periods at capacity have been reached. In the cases that either $x_{i+1} = \varepsilon < D_{i+1}$ or for some period *t*, $x_{(i+1)t} = D_{(i+1)t}$, then there is not a feasible ACC sequence between periods *i* and *j*. As we are assuming non-speculative motives on the cost according to Definition 1, the ACC sequence obtained is of minimum cost. Finally, we note that there is at most only one ACC sequence between any pair of periods. Since we must consider in the worst case *T* periods for determining an ACC sequence and the number of ACC sequences is at most $(T + 1)T/2$ (Florian et al., 1980), the optimal solution of the CLSP can be determined in $O(T^3)$ time by means of the algorithm of Florian and Klein (1971), replacing the procedure for obtaining the production values of the capacity constrained sequences by the procedure described above for the ACC sequences.

Conclusions

In this paper we show that for the CLSP under the assumptions of stationary capacities and concave cost functions with non-speculative motives, i.e. it is profitable to produce as late as possible, the algorithm of Florian and Klein (1971) can be improved from $O(T^4)$ time to $O(T^3)$ time. This result is achieved taking profit the fact that there is an optimal solution that is composed exclusively by a kind of sequences for which the only fractional period, if it exists, is the first among all the positive periods of the sequence. The type of cost structure that we assumed includes many cases of interest. In particular it includes those cases for which the fixed costs of production are no decreasing and non-lineal functions, which according to our best knowledge, are not covered by the algorithms proposed in previous works in the literature.

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