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Optimisation of a strictly concave quadratic form

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Abstract. In this study, we give a method which allows finding the exact optimal solution of a strictly concave quadratic program. The optimisation of a strictly concave program is based on the localisation of the critical point. If the critical point doesn't belong to the feasible solution set the projection onto the hyperplanes passing through the nearest vertex to the critical point gives exactly the optimal solution.

Keywords: concave; convex; projection; separating hyperplane; vertex

Introduction

Let's consider the following quadratic problem P_f:

$$P_f = \begin{cases} x \geq 0 \\ Ax \leq b \\ \max f(x) = \sum_{i=1}^{n} \alpha_i \, x_i + \beta_i x_i^2 \end{cases}$$

The coefficients α_i are any real numbers and $\beta_i < 0$. Let Ω be the convex set formed by the constraints $Ax \leq b$, where A is a $m \times n$ real matrix and b is a vector of $\mathbb{R}_+^m$. Ω is a closed bounded set of $\mathbb{R}^n$. So P_f is strictly concave.

A strictly concave quadratic program can be solved with several methods: the conditional-gradient method, also called the Frank-Wolfe method developed by M. Frank and Ph. Wolfe (1956). It is the most useful method for the nonlinear optimization (see Freund (2004)), the Rosen method (1960). We cite also the Simplex method for quadratic programming detailed by Jensen and Bard (2004).

These methods give only approximate solutions where the difference will be insignificant if the alterations are kept small. Jensen and Bard (2004) and De Wolf (2006) have treated an example with two variables. By applying the Simplex method they have manipulated ten variables. A new algorithm was proposed by Chikhaoui et al (2011) for finding the exact optimal solution without any introduction of any variable. The algorithm is based on the localisation of the local mum $x^* = -\frac{\alpha_i}{2\beta_i}$. If $x^* \in \Omega$ then, it is the global optimum.

Otherwise, the projection of x onto the separating hyperplanes of a new feasible constraint set built by the transformation of Ω gives exactly the global optimum. But, in some cases, this algorithm can give an infeasible solution. In this case, we proof, in this paper, that the optimal solution is the nearest vertex to x^*, and we give the general steps for finding the optimal solution. When the program is not strictly concave, we may always perturb the objective function to obtain a strictly concave problem for any $\varepsilon > 0$.

Projection formula

Let $H = \{x \in \mathbb{R}^n / < a, x >= b\}$, where $a \in \mathbb{R}^n, b \in \mathbb{R}$ are given. Suppose that $x^* \notin H$. That is: $< a, x^* > -b > 0$ or $< a, x^* > -b < 0$.

Let $x_0 \in \mathbb{R}^n$ such that the following equality holds: $x^* - x_0 = ka$, with some positive or negative constant k. The vector $x^* - x_0$ is then parallel to the hyperplane H. For any $x_0 \in H$ the equality $< a, x_0 > = b$ holds. We see that $< a, x^* - ka >= b$ i.e.$< a, x^* > -k. \| a \|^2 = b$, and thus the following relation for k is obtained: $k = \frac{< x^*, a > -b}{\| a \|^2}$ (as $a \neq 0$).

Let $\bar{x} \in H$ in such way that $d(x^*, H) = d(x^*, \bar{x})$. For next, we have $\| x^* - x_0 \| = |k|. \| a \| = \frac{|< x^*, a > -b|}{\| a \|}$, and $\| x^* - x_0 \| \leq d(x^*, \bar{x})$, and thus we hold $\frac{|< x^*, a > -b|}{\| a \|} \geq d(x^*, \bar{x}) = \| x^* - \bar{x} \|$.

As the application $x \longrightarrow < a, x >$, where $a = (a_{11}, a_{12}, \dots, a_{1n})$ defined on $\mathbb{R}^n$ is linear and continuous, then $\| < a, x^* > - < a, \bar{x} > \| = \| < a, x^* > -b \| \leq \| a \| . \| x^* - \bar{x} \|$. So that the following inequality holds $\frac{|< a, x^* > -b|}{\| a \|} \leq \| x^* - \bar{x} \|$ and consequently we obtain $\| x^* - \bar{x} \| = \frac{|< a, x^* > -b|}{\| a \|}$. Thus, $d(x^*, H) = \frac{|< a, x^* > -b|}{\| a \|} = \| x^* - x_0 \| = \| x^* - \bar{x} \|$.

We see then that $\bar{x} = x_0$. Consequently $\bar{x} = x_0 = x^* - ka = x^* - \frac{< x^*, a > -b}{\| a \|^2} . a$. However, the use of this formula doesn't give always a feasible solution (see example 1); we prove in the next, that the optimal solution is either the nearest vertex to the critical point or the projection onto the hyperplanes passing through this vertex.

Condition for a solution to be an extreme point

Let H be any hyperplane that separates x^* and Ω, and let x_0, x_1 be two extreme points in H. The following theorem holds:

Theorem. *If the projection of x^* onto the straight passing through x_0 and x_1 doesn't belong to Ω , then $\bar{x}$ is an extreme point.*

Proof. Let's consider y_0 the projection of x^* onto the straight passing through x_0 and x_1. We know that $< x^* - y_0, z >= 0$ for any z of this straight. For any $u \in [x_0, x_1]$, we have $y_0 - u \perp x^* - y_0$. So $\| x^* - u \|^2 = \| (x^* - y_0) - (u - y_0) \|^2 = \| x^* - y_0 \|^2 + \| u - y_0 \|^2$, and thus $\| x^* - x_1 \|^2 = \| (x^* - y_0) - (x_1 - y_0) \|^2 = \| (x^* - y_0) \|^2 + \| (x_1 - y_0) \|^2$. If $y_0 \notin [x_0, x_1]$, then $\| (u - y_0) \| > \| (x_1 - y_0) \|$, and so $\| (u - y_0) \|^2 > \| (x_1 - y_0) \|^2$. Consequently $\| x^* - u \|^2 > \| x^* - x_1 \|^2$, and $\| x^* - u \| > \| x^* - x_1 \|^2$. For particular $u = x(t) = tx_0 + (1-t)x_1, t \in [0,1]$, we obtain $\| x^* - x_1 \| \leq \| x^* - x(t) \|$, and so $\| x^* - x_1 \| \leq \inf_t \| x^* - x(t) \|$. Consequently $\bar{x} = x_1$.

Algorithm for finding the optimal solution

We give here, the general steps for finding the optimal solution of a strictly concave quadratic program.

1. If $\exists i: \beta_i \neq -1$ then transform the convex;
2. Calculate the critical point $x^* = -\frac{\alpha_i}{2\beta_i}$ for all $i = 1,2, \ldots$;
3. Find the nearest vertex to x^*. Let s be this vertex.
4. Choose a separating hyperplane passing through s then project x^* onto this hyperplane;
5. If the projection gives a feasible solution then the optimal solution is $f(Px^*)$, where Px^* is the projection of x^*;
6. If the projection gives infeasible solutions repeat the steps 4 and 5;
7. If all projections give infeasible solutions, the optimal solution is s.

As a conclusion, the optimal solution of a strictly concave quadratic program can be reached at:

1. an interior point if the critical point belongs to the feasible constraints set;
2. a boundary point if the projection of the critical point onto a separating hyperplane gives a feasible solution;
3. an extreme point in the other cases.

This algorithm is solvable in polynomial time and reaches always his optimum: the convex set is compact, there is, at least, one vertex that is the nearest to x^* which belongs to one separate hyperplane.

Examples

To demonstrate the feasibility of the proposed algorithm, the following examples are solved. The results are compared with those obtained using MINOS 5.51 and CPLEX 11.2.0 (http://www.ampl.com/DOWNLOADS/detail.html, accessed 23 February 2012) used with AMPL Student Edition (http://www.ampl.com/, accessed 23 February 2012).

Example 1

$$\begin{cases} x \geq 0 \\ x_1 + x_2 \leq 2 \\ x_1 + 2x_2 \leq 3 \\ \max f(x_1, x_2) = 14x_1 + 6x_2 \\ -x_1^2 - 2x_2^2 \end{cases}$$

The coefficients $\beta_1 = \beta_2 = -1$, and the critical point $x^* = (7,3) \notin \Omega$. Using the algorithm proposed by Chikhaoui et al (2011) the projections onto all the separating hyperplanes $\boldsymbol{x_1 + x_2 = 2}$ and $\boldsymbol{x_1 + 2x_2 = 3}$ give infeasible solutions. This example is a counterexample of this algorithm. From figure 1, we can easily verify that $s_1 = (2,0)$ is the nearest vertex to x^* . So it is the optimal solution of f. Using the CPLEX, an approximate solution is found using 14 separable QP barrier iterations.

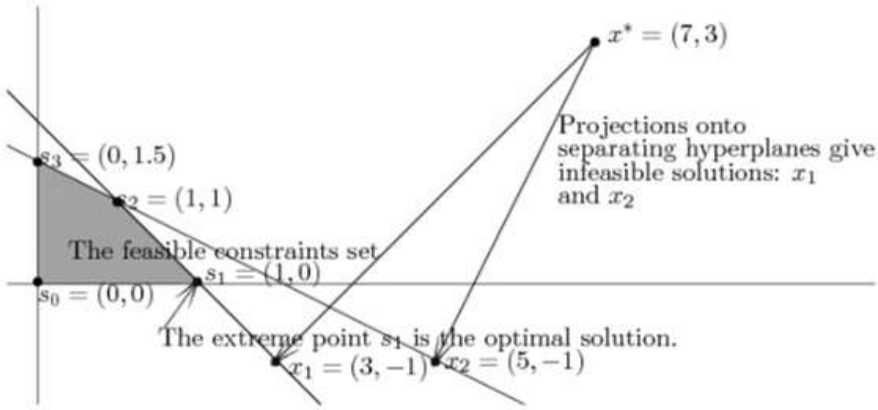

Fig. 1. Projections give infeasible solutions. The optimal solution is the extreme point s_1

Example 2

$$\begin{cases} x \geq 0 \\ x_1 + 2x_2 \leq 8 \\ 3x_1 + 2x_2 \leq 9 \\ \max f(x_1, x_2) = 5x_1 + 5x_2 \\ -x_1^2 - 2x_2^2 \end{cases}$$

The coefficients $\beta_1 = \beta_2 = -1$, and the critical point $x^* = (\frac{5}{2}, \frac{5}{2}) \notin \Omega$. The nearest vertex to $x^*$is $s = (2,3)$. The projection onto the hyperplane $3x_1 + x_2 = 9$ gives a feasible solution $\mathrm{x} = (2.2, 2.4)$. So x is the optimal solution. Using MINOS, we can obtain this solution after 4 iterations. CPLEX gives an approximate solution using 13 separable QP barrier iterations.

Conclusion

In this paper we have given a general steps for finding the exact optimal solution of a strictly concave quadratic program. The algorithm proposed by Chikhaoui et al (2011) allows finding the exact solution in many situations. But, it can give infeasible solutions because the formula given in the precedent section and used by Chikhaoui et al (2011) allows projecting the critical point onto all the separating hyperplanes H and not only on $H \cap \Omega$. Finally, the proposed algorithm can be used for searching the optimal solution of a convex quadratic program: The optimal solution is the farthest vertex to the critical point.

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