Decision boundaries used to model probability of victory and duration of combats

Vesa Kuikka 1,*
1 Finnish Defence Research Agency, Riihimäki, Finland

Abstract—Geometric Brownian motion with decision boundaries is used in modelling combat effects such as probability of winning a battle and duration of a battle. We verify that it is possible to construct a model and analytical mathematical formulas for these quantities. Variables of the model are force sizes at the beginning and at end of a battle, and decision boundaries for the two opposing forces. The model with different model parameters is compared with empirical attrition data. New results can be uncovered when outcomes and expected durations of empirical battles are studied with the model. We conclude that battles, where the attacker is superior to the defender, and battles with comparable force strengths on the two opposing sides, are described with different model parameter values. This can indicate an extra advantage for attackers that cannot be explained by other factors of the model. We show how asymmetrical decision boundaries are used to model this kind of effects. These general conclusions are made on an aggregate level from the modelling results of individual battles.

Introduction

Understanding relationships between the outcome of a battle and factors like force sizes of opposing sides is a fundamental question in military studies. For an army leader to be able to predict combat effects more accurately would be vitally important. Any improvements in tools, methods or understanding would be most welcome. Historically force size is the most important factor but force quality, tactics, casualties, duration, surprise, advance rates, and victory are also important (Hartley, 2001). In modern warfare, technological systems, not only weaponry but also communication and information systems have an increasing importance. The role of information has increased substantially in the last few decades by UAVs and other electromechanical means. Changes are likely to continue, especially in asymmetrical conflicts. Tactics has also changed from historical battles, distributed and networked operation mode has been adopted by many armed forces.

Methods used to study combat effects are numerous and cover many fields of military sciences, information analysis, mathematics, psychology, and social sciences. Much of the work has been heuristic in the sense that results are not presented in quantitative terms. Quantitative methods are usually mathematical formulas or simulation methods. One line in the long history of studying combats is trying to write down an analytic combat equation, as simple as possible, describing combat effects, for example predicting the winner of a battle. However, only few closed form equations exist today. One may argue that no simple formula can describe complex relationships in warfare, and modern warfare is becoming even more complex. In our view, this does not exclude the need to understand the big picture of the complex system of combats. If there are common features in all combats or in in specific types of combats, these features should be recognized and studied with appropriate methods. Secondly, mathematical models and results from the analysis may give a benchmark for comparing and characterizing combats. This would give a basis for more detailed and comprehensive models. Knowledge about which are the most important factors and what are the interrelationships between them is the basic work necessary before a more detailed theory can be developed. Comparing predictions of models and empirical observations provide information about assumptions of the models. Modelling should be realistic enough to enable correct conclusions in these kinds of investigations.

Quantitative equations should be understandable and derived and expressed with real world quantitative terms. In this respect, some combat models are phenomenological and others are based on exactly defined quantities. Examples for the latter are the Lanchester equations (Lanchester, 1914). The Lanchester equations are expressed with force sizes and attrition rates. Examples of phenomenological formulations are the Helmbold relationship (Helmbold, 1989) and the Willard

* Correspondence: Vesa Kuikka, Finnish Defence Research Agency, PO BOX 10, Tykkikentäntie 1. FI-11311 Riihimäki, Finland
E-mail: vesa.kuikka@mil.fi
As a model for attrition processes. The results can be expressed analytically and this is why no simulation boundaries of battles. In the other paper (Kuikka, 2015). Here, we define the stochastic formulations in battles where other factors are unknown. The results show that attackers have an extra advantage which is difficult to explain by other factors in the model. Low variance values or wide decision boundaries can both explain long durations of battles. We show that decision boundaries have a greater effect than different variance values on probabilities and expected durations in battles where attrition rates of the opposing sides are very different. Variance are found to be important in modelling phenomena of even battles and decision boundaries are more important in modelling more extreme battles. Both of these factors can have effects especially on individual battles. In addition, we use decision boundaries for modelling asymmetrical effects between attackers and defenders. The results show that attackers have an extra advantage which is difficult to explain by other factors of the model, i.e. force strengths and variances of attrition processes.

In the next section, we present related work considering analytical macroscopic combat equations. The model of this paper and related work of (Hartley, 2001) have comparable agreement with empirical data. No clear attacker’s advantage has been found in earlier studies (this can also be a consequence of low quality of empirical data). Next, background from earlier work of the author of this paper (Kuikka, 2015) introduces basic concepts. The mathematical formulations for geometric Brownian motion are presented and they can also be found in (Taylor & Karlin, 1998). In this paper a new model for decision boundaries is defined that enables us to predict durations of battles. We can compare, on an aggregate level, theoretical results of probabilities to win battles and expected durations with empirical data.

The empirical attrition data (Hartley, 2001) has no information about variances or decision boundaries of battles. In the model, low variance values or wide decision boundaries can both explain long durations of battles. We show that decision boundaries have a greater effect than different variance values on probabilities and expected durations in battles where attrition rates of the opposing sides are very different. Variance are found to be important in modelling phenomena of even battles and decision boundaries are more important in modelling more extreme battles. Both of these factors can have effects especially on individual battles. In addition, we use decision boundaries for modelling asymmetrical effects between attackers and defenders. The results show that attackers have an extra advantage which is difficult to explain by other factors of the model, i.e. force strengths and variances of attrition processes.
Related work

Combat equations have been studied more than hundred years. The Lanchester equations (Lanchester, 1914) are probably the most known and used deterministic combat equations. The Lanchester equations are solutions to differential equations describing attrition of two fighting troops.

Other closed form combats equations are the Helmbold relationship (Helmbold, 1989; Hartley, 2001) and the Willard equation (Willard, 1962). The Helmbold relation is as follows:

\[
\ln\left(\frac{x_0^2 - x_u^2}{y_0^2 - y_d^2}\right) = c_1 \ln\left(\frac{x_0}{y_0}\right) + c_2,
\]

where \(x_0\) and \(y_0\) are the force sizes of the attacker and the defender at the beginning of a battle. And \(x_u\) and \(y_d\) are the force sizes of the attacker and the defender at end of a battle. In Equation (1) \(c_1\) and \(c_2\) are constants. We use the same notation in Equation (1) as in other equations of this paper. Subscripts \(u\) and \(d\) denote “up” and “down” to describe upper and lower decision boundaries, but in the Helmbold relationship the meaning is more like “attacker” and “defender”.

Extensive research with empirical combat attrition data has been conducted in (Hartley, 2001). The work is based on the Helmbold relation where several factors including force sizes, environmental variables, and human factors have been used as explanatory variables to predict the winner of a battle. The work is mostly heuristic and no modelling from first principles or based on quantitative real world quantities has been done. This is a practical approach taking into account the fact that building a complete theory with so many variables is a difficult task. Some factors can be incorporated in models by including them in force sizes, for example one tank equated as a specific number of soldiers (Dupuy, 1985). Evaluating equivalent numbers of systems, weaponry, and fighters is a complex problem. One method of comparing different sources of military capability and determining equivalent numbers of soldiers and system units is optimizing total capability values (Kuikka, 2016).

Basic Lanchester equations are deterministic. Simulated results based on the Lanchester models have been presented in the literature. Also, stochastic versions of the Lanchester equations and probabilistic models based on Markov chains have been proposed. As a model becomes more complex, solving it analytically becomes intractable or computationally expensive. Approximate solutions have been proposed for computationally intensive problems, such as optimal resource allocation and analysis of asymmetric forces like snipers and fighter aircrafts (Kim et al., 2017; Lappi et al. 2012). Most of these models calculate force sizes and attrition rates while probabilities to win battles and durations of battles are not considered. Deterministic and stochastic models may be extended with the concept of decision boundaries in order to compute these quantities. This line of work is a subject of possible future research. More comprehensive literature review including also stochastic and simulation methods can be found in (Hartley, 2001; Kuikka, 2015).

Background from earlier work

Our earlier paper (Kuikka, 2015) presented a combat model based on geometric Brownian motion with moving decision boundaries. In a way, methods of this paper and earlier work (Kuikka, 2015) can be considered as two manners of approaching the same problem. Attrition processes were modelled as actual force sizes separately for the two opposing forces. Decision boundaries were modelled as linear functions of time. The slope for attacker’s decision boundary was determined by the slope of defender’s force size and vice versa. This model has a property of descending decision boundaries when opposing sides are losing manpower. The probability of winning a battle is expressed in the following form (Kuikka, 2015).

\[
P = \frac{\left(S^{\alpha-1}\right)^{[\beta]} - \left(2S^{\alpha-1} - 1\right)^{[\beta]}}{1 - \left(2S^{\alpha-1}\right)^{[\beta]}}, \text{ where}
\]

\[
S = \frac{x_u y_0}{x_0 y_d} \quad \text{and}
\]

(2)
\[ m = 1 - \frac{2}{\sigma^2}(1 - \alpha)\ln(S). \] (4)

Combat equation (2) is different from combat equations to be presented later in this paper because of different process and decision boundary specifications. In our earlier paper the process and its drift were modelled as the process and its drift of the attrition process itself.

In Equations (2) and (4) \( \alpha \) is a model parameter for adjusting how steep is the decision boundary and \( \sigma \) is volatility (\( \sigma^2 \) is variance in a time unit) of the attrition process. The two bars around \( m \) denote absolute value. In Equation (3) \( x_0, y_0, x_n, \) and \( y_d \) are the initial attacker force strength, initial defender force strength, final attacker force strength, and final defender force strength. To be precise, Equation (2) describes the attacker view of a battle. The corresponding defender view looks much the same and the equation can be found in (Kuikka, 2015) as well as the derivation of the formulas. We will use the quantity \( S \) of Equation (3) throughout in this paper as an independent variable. The quantity \( S \) itself is a good predictor of battle outcomes. If \( S \) is less than one the attacker’s probability to win is less than 50 % and if \( S \) is greater than one the probability is greater than 50 %. However, Equation (3) does not provide numerical probability values as a function of force sizes.

One nice feature of the model is that the equations describing the attacker view and the defender view give the same results if variances on both sides are equal. This can be regarded as a consistency requirement for the model. If the variances are not equal \( \alpha \)-parameters must be different because still the attacker and the defender views should give similar results. This procedure has been explained in (Kuikka, 2015) with examples of the few cases where the variances are actually known from historical data.

In this paper decision boundaries are constants as a function of time. The decision boundary definition is related with the definitions of attrition processes and drift terms. In this paper, drift is defined as the difference of the drift of attacker’s attrition process and the drift of defender’s attrition process. A stochastic process describing the combined attrition process of a battle is approximated with a geometric Brownian process. The new modelling enables predicting both probabilities and durations of battles, not just probabilities as in (Kuikka, 2015). Comparing both of these results, probabilities and durations, with empirical data, can reveal new properties of combat effects. As attrition processes are modelled with geometric Brownian processes, the drift parameter has the standard mathematical definition provided in the literature (https://en.wikipedia.org/wiki/Geometric_Brownian_motion, accessed 25 December 2017; Karlin & Taylor, 1974).

One goal of this paper is to extend the model to predict combat durations and to compare predicted durations with the observed empirical durations of battles. A deficiency of decision boundaries used in (Kuikka, 2015) is that, when \( S \) is close to one in Equation (3), durations became small. This contradicts the observed durations of historical combat data – even though a considerable portion of data points are near (1,0) in \((S, T)\)-coordinates (see Figure 2 later in the text).

We show how the modelling in (Kuikka, 2015) can be carried out with different decision boundaries. A better combat model could be developed with better decision boundary sub-modelling. The problem of constructing more realistic models for attrition processes and decision boundaries still remains a subject for future studies.

**Probability to win a battle and duration of a battle**

We present a variant of our earlier combat equation with new results concerning durations of battles. Constant decision boundaries are stopping boundaries for the combatant party to end a battle. Two decision boundaries for each side are used in the model, for winning and losing a battle. Attrition processes are approximated with geometric Brownian motion describing the difference of attrition processes of the two sides of a conflict. Initial values of the processes are the initial force strengths of attackers and defenders.

The input data for the model are the initial force sizes of the two opposing sides and the corresponding force sizes at the end of a battle. We denote the attacker by \( x \) and the defender by \( y \) because the empirical combat attrition data is available in this form (Hartley, 2001). The theory allows different parameter values for attackers and defenders. The parameters of the model are variances (or mean deviations) of attrition processes and decision boundaries. At the most, two variances and four decision boundaries may exist in the model. On an aggregate level, examining only one or two variance parameter values and one or two values for decision boundaries may be sufficient. The reason is that the empirical data has similarities between attackers and defenders in the view of our model.

Earlier studies (Willard, 1962; Helmhold, 1989; Hartley, 2001) don’t provide direct evidence of an extra advantage for defenders or attackers that cannot be explained by force strengths on an aggregate level. Attackers more often win battles but so far the models have suggested that the superiority of the force sizes can explain the outcome. In this context, we
mean by superiority that defender’s percentage casualties are higher when compared with attacker’s percentage casualties \((S > 1\) in Equation (3)). On the other hand, in many cases the defender is well protected and has a good situational awareness of its own territories. In subsets of the available empirical data, these kinds of advantage factors may be needed in modelling. Individual battles are very diverse and the model should be used with different appropriate parameters for the attacker and the defender when considering an individual battle. In individual battles, surprise can be modelled with different decision boundaries and different variance values for the opposing sides. In summary, conclusions about possible extra advantage factors for attackers or defenders are dependent on details of the used model and specifically how realistic the model is.

We can study the problem on an aggregate level and at individual combat level. The available empirical data has no variance information which should be known or evaluated for considering the results at combat level. In this paper, we model attrition processes at combat level but investigate the results on an aggregate level. A typical variance value for all the battles in the data set is assumed. On an aggregate level, we present typical parameter values with justification for their choices. We compare the results of the model with empirical data.

**Notations**

\begin{align*}
x_0 & \quad \text{Initial force size of attacker} \\
y_0 & \quad \text{Initial force size of defender} \\
x_u & \quad \text{Final force size of attacker}, \ x(T) \\
y_d & \quad \text{Final force size of defender}, \ y(T) \\
A, D & \quad \text{A = Attacker, D = Defender} \\
\mu & \quad \text{Drift value in one day of attrition process (} \mu_A \text{ attacker and } \mu_D \text{ defender), } \mu = \mu_A - \mu_D \\
\sigma^2 & \quad \text{Variance of attrition process in one day (} \sigma_A^2 \text{ attacker and } \sigma_D^2 \text{ defender), } \\
& \quad \sigma^2 = \sigma_A^2 - 2 \text{Cov}(A, D) + \sigma_D^2, \text{ where Cov}(A, D) \text{ is covariance between attacker and defender attrition processes (https://en.wikipedia.org/wiki/Covariance, accessed 21 December 2017)} \\
T & \quad \text{Duration of a battle} \ (T_A = T_D), \ E(T) \text{ expected value of duration} \\
P_A, P_D & \quad \text{Probability to win a battle, } P_A = 1 - P_D \\
\Delta_{su}, \Delta_{sd} & \quad \text{Upper decision boundary is } \Delta_{su} \text{ higher than } x_0 \text{ for attacker, i.e. } X_u = x_0 + \Delta_{su}. \text{ Lower decision boundary is } \Delta_{sd} \text{ lower than } x_0 \text{ for attacker, i.e. } X_d = x_0 - \Delta_{sd}. \\
\Delta_{yu}, \Delta_{yd} & \quad \text{Upper decision boundary is } \Delta_{yu} \text{ higher than } y_0 \text{ for defender, i.e. } Y_u = y_0 + \Delta_{yu}. \text{ Lower decision boundary is } \Delta_{yd} \text{ lower than } y_0 \text{ for defender, i.e. } Y_d = y_0 - \Delta_{yd}. \\
\delta_{su}, \delta_{sd}, \delta_{yu}, \delta_{yd} & \quad \text{Used in the simple model for decision boundaries: } X_u = x_0 + \delta_{su} x_0 = (1 + \delta_{su}) x_0, \quad X_d = (1 - \delta_{sd}) x_0.
\end{align*}

We develop the model in two phases. Before that, the general formulation of the theory is presented. We assume that the attrition processes can be described as geometric Brownian motions (https://en.wikipedia.org/wiki/Geometric_BrownianMotion, accessed 30 October 2016). Two important attributes of the assumption are the stochasticity of the process and the proportionality of the attrition to the force sizes. In the literature (Perry, 2011) has studied geometric Brownian motion in the context of combat modelling. We adopt two results from the literature, the probability to hit the upper stopping boundary and the duration of the process to hit either upper or lower stopping boundaries (Willmott, 2000; http://marcoagd.usuarios.rdc.puc-rio.br/hittingt.html, accessed 30 October 2016). For the purpose of our model these provide the probability to win a battle and the duration of a battle. We use the notations in above. We choose the attacker point of view; the defender view gives comparable results. The relationship between the probabilities holds that the attacker’s probability to win equals the defender’s probability to lose. Upper and lower decision boundaries for the attacker are denoted by \(X_u\) and \(X_d\). The probability for an attacker to win a battle is
The duration of a battle is

\[ E(T) = \frac{1}{\sigma^2} \left( \ln \left( \frac{x_0}{X_d} \right) - \frac{1 - \left( \frac{x_0}{X_d} \right)^m}{1 - \left( \frac{X_u}{X_d} \right)^m} \ln \left( \frac{X_u}{X_d} \right) \right), \]

where \( m = 1 - 2 \frac{\mu}{\sigma^2} \).

Equation (5) is a form of Equation 4.18 in (Taylor & Karlin, 1998) and Equation (6) is from Section 3 Paragraph “Expected First Hitting Time for Either Upper or Lower Boundaries (Geometric Brownian Motion)” in (http://marcoagd.usuarios.rdc.puc-rio.br/hittingt.html). Several sub-models can be created from Equations (5-7) by specifying different models for decision boundaries and variants of attrition processes. Equations (5-7) assume that the stochastic processes describing attrition processes obey geometric Brownian motion. Analytical mathematical results are available for Brownian motion and geometric Brownian motion but for other processes numerical simulations should be used in most cases.

Real attrition processes are descending (with reinforcement ascending processes are possible) and this aspect should be considered at the same time with the specification of decision boundaries. In this paper we study a stochastic process defined as a difference between attacker and defender attrition processes with the drift definition \( \mu = \mu_A - \mu_D = \ln(S) \), where \( S \) is defined in Equation (3). This definition allows constant decision boundaries: \( X_d \leq x_0 \leq Y_u \) and \( Y_d \leq y_0 \leq Y_u \). The end of a battle occurs when attrition processes of the opposing sides hit upper and lower stopping boundaries at the same time \( T \). In the model the winner hits its upper boundary and the loser hits its lower boundary.

Because the stochastic process to be investigated is a difference between two real attrition processes, the variance value is \( \sigma^2 = \sigma_A^2 - 2 \text{Cov}(A,D) + \sigma_D^2 \), where \( \text{Cov}(A,D) \) is covariance between the attacker attrition process and the defender attrition process. Variance is less than the sum of the variances of the two attrition processes. Real attrition processes are positively correlated because casualties of the two sides of a battle are stochastically related with each other.

In the following sections we present two sub-models for the (constant) decision boundaries with the help of available information about attrition processes. In the first model we define the decision boundaries simply as \( X_u = x_0 + d_{ua} (x_0 - x_u) \) and \( X_d = x_0 - d_{ud} x_0 \) for the attacker, and \( Y_u = y_0 + d_{yu} y_0 \) and \( Y_d = y_0 - d_{yd} y_0 \) for the defender. In the second model we define the decision boundaries as \( X_u = x_0 + d_{wa} (x_0 - x_u) \) and \( X_d = x_0 - d_{wd} (x_0 - x_d) \) for the attacker, and \( Y_u = x_0 + d_{ya} (y_0 - y_u) \) and \( Y_d = y_0 - d_{yd} (y_0 - y_d) \) for the defender. Initial force strengths are denoted by \( x_0 \) for the attacker and \( y_0 \) for the defender. Force strengths at the end of the battle are \( x_u \) for the attacker and \( y_d \) for the defender. Numerical values for the force strengths are available from the empirical data. In the formulas the constants \( \delta_{ua}, \delta_{ud}, \delta_{wa}, d_{wa}, d_{wd}, d_{ya}, d_{yd} \) are parameters of the models. If the attacker wins \( X_d \) and \( Y_u \) describe the boundaries that are not directly observed. If the defender wins \( X_u \) and \( Y_d \) describe the boundaries that are not directly observed.

There is a relationship between decision boundaries and variances: low variances or wide decision boundaries and high variances or narrow decision boundaries provide similar results for expected durations. Numerical values for decision boundaries and variances are not available from the empirical data (Hartley, 2001). In the second model we vary decision boundary values and use typical variance values to compare the results of the model and the empirical data. However, in the next section we experiment with different variance values and decision boundaries to get some insight about their typical range.
Model 1

In this section, we present a simplified model to demonstrate the effects of different parameter values. We show how the duration of a battle changes with variance values, with different decision boundaries and with asymmetrical decision boundaries. In this introductory model decision boundaries are $\delta x_0$ above and below the initial force size value $x_0$, where $\delta$ is a constant. If $\delta = \delta_u = \delta_d$ ($\Delta_u = x_0 + \delta x_0, \Delta_d = x_0 - \delta x_0$) and the upper decision boundary is 10% above $x_0$ and the lower decision boundary is 10% below $x_0$, we get from Equations (5) and (6) for the probability to win

$$P = \left( \frac{x_0}{x_0 + \delta x_0} \right)^m \left( 1 - \left( \frac{x_0 - \delta x_0}{x_0} \right)^m \right) = \frac{1}{1 + \delta} \left( 1 - \delta \right)^m = \frac{1}{1.1} \left( 1 - (0.9)^m \right),$$

(8)

and for expected duration

$$E(T) = \frac{1}{\sigma^2} \left( \ln \left( \frac{1}{1 - \delta} \right) - \frac{1}{(1 - \delta)^m} \ln \left( \frac{1 + \delta}{1 - \delta} \right) \right)$$

$$\frac{1}{\sigma^2} \left( - \ln(0.9) - \frac{1}{(1 - (0.9)^m)} \ln \left( \frac{1.1}{0.9} \right) \right), \text{ where}$$

$$m = 1 - \frac{\ln(S)}{\sigma^2} \text{ and } S = \frac{x_u y_0}{x_0 y_d} = \frac{x_0 + \delta x_0}{x_0 y_d} = \frac{1.1 y_0}{y_d}.$$  

(9)

With Equations (8) and (9) we can show some typical effects of the parameters $\sigma^2$ and $\delta$. Equations (8) and (9) are for symmetrical decision boundaries. Equations for asymmetrical decision boundaries can be expressed in a comparable manner. The effects on the expected duration value of a battle are investigated as a function of $S$ in Equation (9) (equivalently as a function of $y_0 / y_d$ due to the definition of $m$). In Figure 1A four curves with $\sigma^2 = 1/100, 1/200, 1/300$ and $1/400$ are shown. Near $S=1$ ($y_d = 1.1 y_0$) values of $E(T)$ are higher for lower values of $\sigma^2$. This is a direct consequence of the functional form of Equation (9). This effect prevails only when $\sigma^2$ is approximately between 0.8 and 1.25. Outside this interval $E(T)$ is almost independent of variance value. When $S=1$ the value of $E(T)$ is proportional to $1/\sigma^2$. The expected duration values of a battle are 1, 2, 3, and, 4 days for the variance values mentioned earlier. The interpretation is obvious: higher variation means heavier fighting and shorter duration of a battle.

In Figure 1B a similar comparison with wider decision boundaries is shown. The variance value of $\sigma^2 = 0.01$ is used and the effects of decision boundaries symmetrically 10%, 15% and 20% above and below the initial value of force size $x_0$ are shown. With wider decision boundaries $E(T)$ is higher. This occurs with all values of $S$ in contrast to the curves shown in Figure 1A. The interpretation is that the adversary is more resilient and is not ready to surrender. The battle takes a longer time and the higher expectation time for duration follows.

In Figure 1C an interesting phenomenon can be seen when the decision boundaries are not symmetrical. In this case the upper decision boundary is higher than the corresponding symmetrical upper decision boundary with respect to $x_0$, would be. In Figure 1C $\delta_u = 0.1$ is the symmetrical case and $\delta_u = 0.2, 0.3$ and 0.4 are asymmetrical results. The value for the lower boundary $\delta_d = 0.1$ is used in all the cases. The consequence of non-symmetry is asymmetrical behavior also in $E(T)$. With high values of $S$ the expected duration value of a battle is higher when compared with the symmetrical case. It is noticeable that effects with low values of $S$ are negligible. The interpretation is that when $S > 1$ the attacker is superior.
and probably closer to the victory, that is, closer to the upper decision boundary and changes on the upper boundary are more important than changes on the lower boundary far away down. The situation is reversed when $S < 1$.

The behavior of the expected duration values in the three situations of Figure 1 can be used in reasoning whether a consistent model of combat can be constructed which have force sizes, variances, decision boundaries and durations as explanatory factors or predicted values.

**Figure 1.** Effects on expected combat durations from Equation (9). A) Different variances, B) Different decision boundaries, C) Asymmetrical decision boundaries.

**Model 2**

We proceed with a model of more detailed definition for decision boundaries. Decision boundaries depending on a particular condition of a battle may predict outcomes of battles better. From Equations (5-6) with $\mu = \ln(S)$ the probability to win a battle and expected duration can be expressed as
In this model decision boundaries are defined with the help of casualties. The upper and lower decision boundaries for the attacker and the defender are defined as

$$X_u = x_0 + d_{uw} (x_0 - x_u), \quad X_d = x_0 - d_{ud} (x_0 - x_u),$$

and

$$Y_u = y_0 + d_{yu} (y_0 - y_u), \quad Y_d = y_0 - d_{yd} (y_0 - y_d).$$

(12)

Decision boundaries are calculated for each battle with their individual force size values. In attacker view we need $X_u$ and $X_d$, and in defender view $Y_u$ and $Y_d$. The attacker view and the defender view give comparable results but not exactly the same. Because both views must give similar results, different parameter values for individual battles should be used, but these empirical values are not available at combat level. To avoid this difference we take the attacker view when $S > 1$ and the defender view when $S < 1$. This makes the calculations more consistent and symmetrical with respect to $S$. The corresponding choice is applied also for expected durations. The knowledge about the winner of a battle is not used. When $S = 1$ we can use either the attacker view or the defender view. The model of Equation (12) for decision boundaries gives the probability to win a battle as

$$P = \frac{\left( x_0 \right)^m - \left( x_0 - d_{uw} (x_0 - x_u) \right)^m}{\left( x_0 + d_{uw} (x_0 - x_u) \right)^m - \left( x_0 - d_{uw} (x_0 - x_u) \right)^m} = \frac{x_0}{x_0 + d_{uw} (x_0 - x_u)}^{\frac{2 \ln(S)}{\sigma^2}} \left( 1 - \frac{x_0 - d_{uw} (x_0 - x_u)}{x_0} \right)^{\frac{2 \ln(S)}{\sigma^2}}, \quad \text{when } S \geq 1,$$

(13)

$$P = 1 - \frac{\left( y_0 \right)^m - \left( y_0 - d_{yd} (y_0 - y_d) \right)^m}{\left( y_0 + d_{yd} (y_0 - y_d) \right)^m - \left( y_0 - d_{yd} (y_0 - y_d) \right)^m} = \frac{y_0}{y_0 + d_{yd} (y_0 - y_d)}^{\frac{2 \ln(S)}{\sigma^2}} \left( 1 - \frac{y_0 + d_{yd} (y_0 - y_d)}{y_0} \right)^{\frac{2 \ln(S)}{\sigma^2}}, \quad \text{when } S < 1.$$

(14)

For expected duration we have

$$E(T) = \frac{1}{\sigma^2} \left[ \ln \left( \frac{x_0}{x_0 - d_{ad} (x_0 - x_u)} \right) - \frac{1}{\sigma^2} \left( \frac{x_0}{x_0 + d_{ad} (x_0 - x_u)} \right)^{\frac{2 \ln(S)}{\sigma^2}} \ln \left( \frac{x_0 + d_{ad} (x_0 - x_u)}{x_0 - d_{ad} (x_0 - x_u)} \right) \right],$$

(15)
when $S \geq 1$, and

$$E(T) = \frac{1}{\sigma^2} \ln\left(\frac{y_0}{y_0 - d_{yd}(y_0 - y_d)}\right) - 1 - \left(\frac{y_0 - d_{yd}(y_0 - y_d)}{y_0 - d_{yd}(y_0 - y_d)}\right)^{1 + \frac{2\ln(S)}{\sigma^2}} \cdot \ln\left(\frac{y_0 + d_{yd}(y_0 - y_d)}{y_0 - d_{yd}(y_0 - y_d)}\right), \quad (16)$$

when $S < 1$

In this approach decision boundaries are constant which is related to the definition of drift term of a geometric Brownian motion describing the combined attrition process of an attacker and a defender.

**Comparison with the empirical data**

In this section we compare the results of the model presented in the previous section with the empirical combat data (Hartley, 2001). The quality of the empirical data is not high and also the data is scarce for low and high values of $S \notin (0.9, 1.1)$. Also a number of battles have very long duration or very extremely high casualties. As said before, the same durations can be achieved with high variance values or narrow decision boundaries and low variance values or wide decision boundaries. As consequence, it is difficult to fit the parameters of the model with the empirical data. In the following we compare the theoretical results, probabilities and durations, with the empirical data and try to find typical parameter values that are consistent with the theory.

We proceed in three steps. First, we compute the distribution of durations as a function of $S$ and make conclusions which values of variance agree with the empirical data on an aggregate level. On the basis of the first step, we choose one value for the variance and set all the four parameter values for decision boundaries as one. In the third step, we adjust the model parameters, variances and decision boundaries, to find a set of parameters that can describe most of the empirical data. Parameters for individual battles can vary a lot, and in modeling different variances and decision boundaries should be used for each battle. Our goal is to study macroscopic combat effects on an aggregate level. We have two research questions: Is it possible to find a set of model parameter values that describe aggregate empirical results? If yes, can we make conclusions about characteristics of general combat effects? First, we investigate whether distribution of empirical and theoretical duration values of Model 2 agree with each other. In Figure 4A the probabilities from Equations (13-14) for attackers to win a battle are shown for the empirical data points have been calculated from theoretical Equations (15-16) for the three different variance values $\sigma^2 = 0.0025$, $\sigma^2 = 0.01$ and $\sigma^2 = 0.025$. Here, we have set the decision boundary parameters as $d_{uu} = d_{ud} = d_{uy} = d_{yd} = 1.0$. Figure 3 shows the corresponding empirical results (Hartley, 2001) as a function of $S$. Comparing Figure 2 with Figure 3 an estimation of a typical variance value matching the distributions of aggregate level data can be made.

In Figure 4A the probabilities from Equations (13-14) for attackers to win a battle are shown for the empirical data points with variance $\sigma^2 = 0.0025$ and symmetrical decision boundaries of parameter values $d_{uu} = d_{ud} = d_{uy} = d_{yd} = 1.0$. In Figure 4A the dotted curve shows the probability results calculated with an empirical formula in (Hartley, 2001). To have a check the solid curve shows the same probabilities calculated directly from the empirical data set. These two curves are close to each other which indicate that the empirical data is correctly represented in Figure 4A. The curves also provide rough error estimation for the empirical data.

Figure 4A shows discrepancy of probabilities between the model and the empirical data. With the values of $S \in (0.9, 0.0)$ the model and the data agree at an average level but with values $S \in (0.0, 1.05)$ the model predicts too low values for the probabilities to win a battle. This would indicate that attacker and defender sides are not symmetrical with each other like some earlier studies have indicated (Willard, 1962; Helmbold, 1989; Hartley, 2001).

Figures 4B and 4C show durations as a function of $S$ when $S \in (0.9, 1.1)$ and $S \in (0.5, 2.0)$ respectively. The dotted curves show the empirical observations and the solid curves show the values calculated from Equations (15-16). When $S \in (0.9, 1.1)$ the theoretical curves are above the empirical curves. A higher variance would give somewhat better results but that would make the probabilities in Figure 4A less compatible with the empirical values. Taking into account high uncertainty of the empirical data, the variance value of $\sigma^2 = 0.0025$ can be regarded as a rough compromise between Figures 4A and 4B.
Figure 2. Expected durations from Equations (15-16) with $d_{uu} = d_{zd} = d_{yu} = 1.0$ and different values of variance $\sigma^2$. A) $\sigma^2 = 0.0025$, B) $\sigma^2 = 0.01$, and C) $\sigma^2 = 0.025$.

Figure 3. Empirical durations directly from the data set published in (Hartley, 2001) as a function of $S$. 
Figure 4C shows the same quantities as Figure 4B at a wider interval. At higher values of $S$ the theoretical curve is usually below the empirical curve. The interpretation is that when $S \gg 1$ the defender is more resilient, in the sense of not surrendering, when compared with battles when $S \approx 1$. The same phenomenon can be observed when $S \ll 1$, but it is not so apparent, and also less empirical data is available than for high values of $S$. Lower variance for $S \gg 1$ than for $S \approx 1$ is the second possible factor that could explain the discrepancy between the model and the empirical data. We conclude from Figure 1A of Model 1 that different values of variance $\sigma^2$ provide almost similar results for low and high values of $S$. We assume that on an aggregate level, the explaining factors are mostly decision boundaries rather than variances (at a detailed level of individual battles this generalization is not valid).

Figure 4C illustrates that the variance value of $\sigma^2 = 0.0025$ together with the symmetrical decision boundaries parameters $d_{yu} = d_{yd} = d_{yu} = d_{yd} = 1.0$, are not appropriate when $S > 1.05$. The model parameter values in Figure 4 serve as a starting point for determining more appropriate parameter values explaining both probabilities and expected durations as a function of $S$.

**Figure 4.** Results for $\sigma^2 = 0.0025$ and symmetrical decision boundary parameter values $d_{yu} = d_{yd} = d_{yu} = d_{yd} = 1.0$. A) Dots show theoretical probabilities to win a battle. Solid curve shows the empirical data from (Hartley, 2001) and dotted curve shows values of an empirical formula in (Hartley, 2001), B) Solid curve shows theoretical expected durations and dotted curve shows the average empirical data, C) As in B for a wider interval of $S$. 

Figures 5 shows the corresponding results with symmetrical narrower decision boundaries \( d_{yu} = d_{xd} = d_{yu} = d_{yd} = 0.8 \). This makes the duration results in Figure 5B more compliant with the empirical values for \( S \in (0.94, 1.06) \). Still, the empirical and theoretical values are different for probabilities in Figure 5A when \( S \in (0.0, 1.05) \) and for expected durations in Figure 5C when \( S > 1.3 \). In other words, distinct intervals of \( S \) may exist where different model parameters should be used. A fact should be taken into account when the empirical data set is used: Durations have been documented as integer values. This may result in biased too high values in empirical data.

\[ \begin{align*}
\text{A: Probability (}\sigma^2=0.0025) \\
\text{B: Expected duration (}\sigma^2=0.0025) \\
\text{C: Expected duration (}\sigma^2=0.0025)
\end{align*} \]

Figure 5. Corresponding results as in Figure 4 for variance \( \sigma^2 = 0.0025 \) and decision boundaries 20% closer.

Now we apply the idea of Figure 1C and use asymmetrical decision boundaries to lower the theoretical curve when \( S >> 1 \). Figures 6A and 6B show the results with the variance value of \( \sigma^2 = 0.005 \) and with the asymmetrical decision boundary parameters of \( d_{uu}=1.0, d_{sd}=2.0, d_{yu}=0.8, d_{yd}=1.0 \) The value of variance is a little higher than in Figure 4. The lower decision boundary values for attackers have been lowered two times lower with respect to the corresponding symmetrical values. Now the empirical and theoretical probability values in Figure 6A are closer to each other in the intermediate interval of \( S \in (0.9,1.1) \). The discontinuity in Figure 6A is irrelevant when we consider \( S \) values not very close to one. At the same time, the theoretical expected duration values are also more consistent with the empirical data. However, at low and high values of \( S \notin (0.7,1.4) \) the same discrepancy as in Figure 4 still exists in Figure 6B.
Figure 6. Results with $\sigma^2 = 0.005$ and different decision boundaries. A) Probabilities with asymmetrical decision boundaries $d_{su} = 1.0$, $d_{sd} = 2.0$, $d_{su} = 0.8$, $d_{sd} = 1.0$, B) Expected durations with asymmetrical decision boundaries as in A with $d_{su} = 1.0$, $d_{sd} = 2.0$, $d_{su} = 0.8$, $d_{sd} = 1.0$, C) Expected durations with decision boundaries $d_{su} = d_{sd} = 2.0$, $d_{su} = 0.8$, $d_{sd} = 1.0$ for $S > 1.2$.

Figure 6C shows results with the variance value of $\sigma^2 = 0.005$ and even wider decision boundaries with the parameters of $d_{su} = d_{sd} = 2.0$, $d_{su} = 0.8$, $d_{sd} = 1.0$. The parameter values are symmetrical for attacker decision boundaries as there is no direct evidence for asymmetrical decision boundaries for high values of $S > 1.3$. As the empirical data is scarce and diverse for these extreme events, it is not possible to make definite conclusions about possible asymmetry. However, wider decision boundaries can explain longer expected durations.

Summary

We have constructed a stochastic model describing combat effects based on geometric Brownian motion and decision boundaries. Geometric Brownian motion describes force sizes and decision boundaries describe decision rules for ending a battle. The model uses constant decision boundaries. In the terminology of stochastic analysis decision boundaries are called stopping boundaries. The stochastic process of a battle is defined as a difference between the attacker attrition process and the defender attrition process.
Parameters of the model describing a battle are variance and two decision boundaries for both of the opposing sides. The model allows different parameter values for attackers and defenders. Modelling is done at detailed level of individual battles but results are studied on an aggregate level. This is because the available empirical database has no data about variances or decision rules for individual battles. Our goal was to investigate general characteristics of combats by comparing theoretical results and empirical attrition data.

Input variables of the model are the force strengths at the beginning and at the end of a battle. In the model higher variance values and wider decision boundaries are needed when one of the two forces is superior to the other. The interpretation is that on an aggregate level intense fighting, higher variance and resilient defenders are characteristic (or at least some of them) features in majority of these battles. In extreme battles, decision boundaries are more important modelling factors than variances in explaining longer durations of empirical data.

In addition, the model suggests asymmetrical decision boundaries when attackers are superior to defenders. This indicates that attackers have an extra advantage with respect to defenders. Asymmetrical features between attackers and defenders are discovered by comparing empirical data with the modelling results of probabilities and durations.

In this paper, analytical closed form formulas are presented for calculating the probability of victory and the value of expected duration of a battle. The probability to win a battle or a war is a combination of numerous different decisions on many levels of decision making. The modelling methods can be used for developing more detailed sub-models for decision rules. These techniques can have applications in decision making situations in other contexts than military.

References