A risk incentive problem for a lender and a borrower when an additional asset is used as security

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Abstract. A typical problem in financial contracting is the so-called risk-shifting incentive problem. In many cases, this problem in financial contracting, which typically arises in the context of a lender-borrower relationship, means the borrower’s incentive to influence the risk of his project. In particular, the shareholders of a levered firm will behave so as to maximize the value of their shares since the shareholders’ objectives do not necessarily coincide with firm value maximization when the firm is levered. However, a lender also may have a risk incentive under some circumstances; thus, he may make it difficult to carry out efficient debt renegotiation. We study not only the borrower’s risk incentive but also the lender’s risk incentive, and the relationship between both sides is studied since the lender’s risk incentive may generate the borrower’s risk incentive. Thus, when either side selects a risky project instead of a safe project, inefficiencies in resource allocation develop. In this paper, we focus on an effective financing strategy of a lender for which lender’s expected profit is valued by taking borrower’s risk incentive into account. The possibility of moral hazard raised by borrower’s firm is considered as a cause which the borrower himself may make agency cost high. In particular, when the lender perceives the possibility of risk incentive of the borrower he may impose to require more additional amount to be paid since the lender gets afraid of involving in a financial trouble. Then, the borrower will refrain from investing in the risky project, instead, he may use an additional asset as security to be able to maintain or recover his trust or credibility. Thus, it is shown that the existence of the additional asset will affect the borrower’s incentive. Through numerical simulations, we analyze the risk-shifting incentive problem, then the agency cost generated with the effect of an additional security is discussed, as a result the risk incentive problem may be mitigated. For our approach to analyse the process, option pricing theory is used and examined with some examples.

Keywords: barrier option; security; shareholder and creditor; option pricing; risk-shifting incentive

Introduction

When considering moral hazard in a financial contract, some asymmetric information may exist between a principal (financial intermediary) as a lender and an agent as a borrower (company). For example, when the behavior of a borrower is not perceived by a lender, the borrower may break the contract because he will have an incentive to invest in a project that gives a higher return with greater risk after borrowing the money. This problem is the so-called risk-shifting incentive problem and was first studied by Jensen and Meckling (1976). Such incentive problems inflict a loss on lender who is at a disadvantage, and produce inefficient resource allocation. In a serious situation, a lender may not conclude a contract with a borrower, causing financial activity and the market to decline. To solve the
above incentive problem, it is advisable for the lender to give the borrower some incentive that will encourage the borrower not to behave dishonestly or act in an undesirable manner when the lender and borrower have a contract, resulting in a fine principal-agent relationship. In other words, the lender is required to lend normally to the borrower, who behaves honestly. If a contract with an appropriate incentive can be designed, the problem generated by asymmetric information will be reduced, possibly leading to the more efficient allocation of resources and greater social efficiency. Thus, the type of incentive in the contract that is suitable and its optimal design are the key to solving the asymmetric information problem.

There are several ways to resolve or improve the above situation. Chesney and Gibson-Asner (2001), Green (2001) and Chiesa (1992) argue that debt with warrants or convertible debt can realign the incentives of bondholders and shareholders. Smith and Warner (1979) and Kalay (1982) mention the role of the provisions of contracts. Others including Gertner and Scharfstein (1991) point out that debt renegotiation often cannot be agreed due to conflicts among these creditors when the borrowing firm has a number of different creditors. As a means of mitigating risk-shifting incentive problems in firms, Ziegler (2004) focuses on the strategy of the borrower and discusses the borrower’s risk incentive. He points out that providing an additional asset for security is an effective way of reducing the incentive problem in terms of more efficient resource allocation. Moreover, Isagawa (2006) focuses on the strategy of the lender and discusses incentive problems between the lender and the borrower. They point out that the debt concession of the lender is an effective way of reducing the incentive problem. In general, the shareholders of a levered firm will behave so as to maximize the value of their shares since the shareholders’ objectives do not necessarily coincide with firm value maximization when the firm is levered. If the lender also has a risk incentive under some circumstances, thus, he may make it difficult to carry out efficient debt renegotiation. Thus, when either side selects a risky project instead of a safe project, inefficiencies in resource allocation develop.

In this study, we focus on an effective financing strategy of a lender for which lender’s expected profit is valued by taking borrower’s risk incentive into account. The possibility of moral hazard raised by borrower’s firm is considered as a cause which the borrower himself may make agency cost high. In particular, when the lender perceives the possibility of risk incentive of the borrower he may impose to require more additional amount to be paid since the lender gets afraid of involving in a financial trouble. Then, the borrower will refrain from investing in the risky project, instead, he may use an additional asset as security to be able to maintain or recover his trust or credibility. Here, as mentioned in Chan and Kanatas (1985), we use an additional asset that would normally not be available to the lender. It is shown that the existence of the additional asset will affect the borrower’s incentive. Through numerical simulations, we analyze the risk-shifting incentive problem, then the agency cost generated with the effect of an additional security is discussed, as a result the risk incentive problem may be mitigated. As an approach to analyze this problem, we consider it in a framework of option pricing theory, in which a lending bank lends money to a borrowing firm for a finite time period. We note that for the option pricing formula like Black-Scholes (B-S model) some barriers must be attached to the formula, since in the standard B-S valuation model shareholders are supposed to always select infinite-volatility projects. Our results are examined and illustrated with some examples.

The paper is organized as follows. In chapter 2, the structure of contract between a lender and a borrower is described, and how option pricing theory, which can be applied and fitted into a risk incentive problem of both sides, is explained. Chapter 3 talks about a risk incentive problem for a lender and a borrower, for which the payoffs for the borrower and lender with its optimal volatilities are described. Also, a safe project and risky project are introduced with some example. In chapter 4, the effect of an additional asset as security is discussed in which some comparison with the results of chapter 3 is illustrated by valuing agency costs. Finally, concluding remarks are stated in chapter 5.

**Structure of contract**

It is well known that MM theory proposed by Modigliani and Miller says, in general, that the value of a firm has nothing to do with the structure of capital. This theory is developed on the base of the assumption of perfect competition market without the both tax and trade costs. On the other hand, as mentioned in Chesney and Gibson-Asner (2001), once the debt is issued, the investment policy of the firm is the only decision controlled by shareholders. Also, the value of the assets depends on the investment policy chosen by the shareholders through its impact on the firm’s return volatility parameter $\sigma$. 

Thus, we assume that the shareholders of a levered firm will act so as to maximize the value of their shares. Then, if the strategic behavior on behalf of the shareholders is invoked, they will choose an optimal volatility for the firm that maximizes the value of their shares.

As we mentioned before, we are concerned in discussing a risk incentive problem in a framework of option pricing theory. In this recognition, the seminal work of Black-Scholes and Merton may give an alternative consideration to value various claims written on the value of the firm, which is used to explain the wealth-maximizing conflicts of interest between a lender and borrowers (shareholders).

We consider a lending bank (a lender) and a firm (a borrower) that reach the following agreement. We assume that all projects have a finite lifetime $T$. Here, we treat a simple case without considering interest rates. Our main steps taken for concerning the contract will follow below.

1. At initial time $t$, a lending bank lends money $D(=I)$ to a borrower in exchange for a promise by the borrower to pay him $D = D\exp(r_d(T-t))$ at loan maturity time $T$. $r_d$ is interest rate to lend the money to the borrower.

2. Then, the borrower invests the money after receiving it from the lender. The borrower selects an asset among many possible investment projects. At the initial time the values of all assets are equivalent and the risk $\sigma_i$ ($i=1...n$) is different for all assets. Thus, at time $\tau$ ($t \leq \tau \leq T$) the borrower can choose to invest all his funds in any one of a series of projects $i=1...n$, whose value dynamics $S_{i,\tau}$.

Now, we turn to some discussion related to the simple and basic concept on which Black-Scholes formula in applying option pricing theory can be derived. First of all, our basic assumption, common to most of option pricing theory, is that we don’t know and can’t predict tomorrow’s values of asset prices. The past history of the asset value as a financial time series is there to examine as much as we want, but we can’t use it to forecast the next move that the asset will make. This does not mean that it tells us nothing. We know from our examination of the past what are the likely jumps in asset price, what are their mean and variance and, what is the likely distribution of future asset prices. Almost all models of option pricing are founded on one simple model for asset price movements, involving parameters derived from historical or market data. According Wilmott et al. (1995), it is often stated that asset prices must move randomly, which is called as random walk, because of the efficient market hypothesis. This hypothesis may be strong assumption, which is several different forms, but they say basically two things: (1) The past history is fully reflected in the present price, which does not hold any further information. (2) Markets respond immediately to any new information about an asset. With these two assumptions above unanticipated changes in the asset price are a Markov process. In this recognition, the commonest model for the corresponding return on the asset decomposes into two parts. One is a predictable, deterministic and anticipated return akin to the return on money invested in a risk-free bank. The other allows the random change in the asset price in response to external effects such as unexpected news. Thus, we assume the dynamics of $S_{i,\tau}$ in our case follow the following stochastic process,

$$dS_{i,\tau} = rS_{i,\tau}dt + \sigma_iS_{i,\tau}dW_t \quad (i=1\cdots n),$$

where $r$ is the risk-free rate and $dW_t$ denotes standard Brownian motion. The borrower can, at any time, switch to another project with a different risk after investing in a project.

3. We assume that all assets have the same maturity time $T$ and that the final return $S(\sigma,T)$ can be observed by both the lender and the borrower at time $T$. If the value of the asset drops to around $L(<D)$, which is fixed, then the lender perceives the situation, the project is immediately suspended and the contract between the lender and borrower is terminated. At that time, the lender can obtain the asset of value $L$, and the borrower’s profit becomes 0 because the asset is withdrawn to pay the lender. In this case, $L$ is interpreted as follows. If the asset value reaches $L$, the firm’s financial health is too weak to allow for additional equity financing. In other words, the lower bound $L$ can be viewed as a critical maintenance level of the assets necessary to secure future debt payments. Also, this is called a knock-out condition which is explained later.

4. At maturity time $T$ the borrower pays $V_{Lender}(\sigma,D,T)$ back to the lender, and the payoff of the borrower is $S(\sigma,T) - V_{Lender}(\sigma,D,T)$. Also, we assume that the lender and borrower have limited liability.
Risk-shifting incentive problem for lender and borrower

Financial contract and option theory

The stakeholders in a financial contract are the lender and borrower. In general, it is considered that the payoff structure of the borrower implies a long position for the call option, for which the assets of the project are considered as the underlying assets and the loan payment is considered as the strike price. On the other hand, the lender is in the short position for the whole value of the assets and the call option. Therefore, since the downside risk for the borrower is limited and, at the same time, profit is not limited for the upside, the borrower may have an incentive to invest in a project involving higher risk.

In applying option pricing theory to the payoff structure, as the volatility in the B-S formula increases, the value of the call option possessed by the borrower increases. In contrast, while the lender cannot receive profit for the upside when a highly risky project is successful, he also receives a loss for the downside when the project fails. This implies that the lender, who has the short position for the call option, can increase his value by controlling the volatility. However, some caution in using the B-S model must be recognized in the light of volatility as mentioned in the introduction. In fact, Chesney and Gibson-Asner (2001) point out that in the standard B-S equity valuation model, shareholders always select infinite volatility projects. Smith and Warner (1979) and Green (2001) claim that standard option pricing modeling along B-S lines can be taken only as a rough approximation because it does not recognize that the value of the firm is endogenous.

Therefore, in this study we pay attention to taking the dynamics of the asset value of a project before maturity into account. The reason for this is that, when the asset value reaches a default value $L$ before maturity $T$, the lender may recognize the fact, and hence he can suspend the project immediately. In general, although the lender cannot monitor the details of the asset value, it is not considered realistic for a manager to allow the behaviour of the borrower to take its own course until the value of the loan that can be collected becomes 0. Thus, in financial contracts it is often observed that the contract is forcibly suspended before maturity and the asset is seized.

Borrower’s payoff and optimal volatility

We denote the profit of the borrower at maturity time $T$ by

$$V_{\text{Borrower}}(\sigma, D, T) = v(\sigma, D, T).$$

(2)

At the time of loan maturity, the profit of the borrower is the remainder after the promised loan $D$ is subtracted from the asset value $S_T$. However, the borrower’s profit does not become negative even if default occurs. On the other hand, the profit for the borrower becomes 0 if the asset value falls below $L$ only once before time $T$, because a knock-out condition that considers the fluctuation of assets before maturity is included. Therefore, the payoff structure of $v(\sigma, D, T)$ implies a long position for the down-out call-type knock-out option, for which the assets of the project are considered as the underlying assets and the loan payment is considered as the strike price. According to Chesney and Gibson-Asner (2001), $v(\sigma, D, T)$ can be expressed as follows:

$$v(\sigma, D, t) = \begin{cases} \max[S_T - D, 0] & \min(S_T) > L \\ 0 & \text{otherwise} \end{cases}$$

$$= S_T e^{(\text{r}(T-t))} \Phi(d_1) - D\Phi(d_1 - \sigma\sqrt{T-t}) - S_T \left( \frac{S_T}{L} \right)^{-\frac{\sigma}{\sigma^2+1}} \Phi(d_2)$$

$$+ \left( \frac{S_T}{L} \right)^{-\frac{\sigma}{\sigma^2+1}} D\Phi(d_2 - \sigma\sqrt{T-t})$$

(3)

where $d_1 = \frac{\ln(S_T / D) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = \frac{\ln(L^2 / S_T D) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, and $\Phi(\cdot)$ is the cumulative probability density function of the standard normal distribution.
Note that the equation for the profit of the borrower still involves the barrier \( L ( < D ) \). Then, as stated before, if the shareholders of a levered firm behave so as to maximize the value of their shares, we look for an optimal volatility level. As the similar in Chesney and Gibson-Asner, the derivative of the profit of the borrower \( \frac{\partial V_{\text{Borrower}}(\sigma, \bar{D}, T)}{\partial \sigma} \) with respect to \( \sigma \) is

\[
\frac{\partial V_{\text{Borrower}}(\sigma, \bar{D}, T)}{\partial \sigma} = \frac{S_T e^{r(T-t)}}{\sqrt{2\pi}} \left[ \exp \left( -\frac{d_1^2}{2} \right) - \left( \frac{S_t}{L} \right)^{-\frac{\sigma^2 t}{2}} \exp \left( -\frac{d_2^2}{2} \right) \right] \\
- \frac{4r}{\sigma^3} \ln \left( \frac{S_t}{L} \right) \left( \frac{S_t}{L} \right)^{-\frac{\sigma^2 t}{2}} \left( S_T e^{r(T-t)} \Phi(d_2) - \bar{D} \left( \frac{S_t}{L} \right)^2 \Phi(d_2 - \sigma \sqrt{T-t}) \right).
\]

Thus, the optimal risk \( \sigma^*_\text{Borrower} \) that maximizes the borrower’s profit is found at the default boundary \( L \) where

\[
\frac{\partial V_{\text{Borrower}}(\sigma, \bar{D}, T)}{\partial \sigma} \bigg|_{\sigma=\sigma^*_\text{Borrower}} = 0.
\]

We solve (5) later numerically by showing some example because of the absence of a known closed-form solution.

**Lender’s payoff and optimal risk**

The possible loan collected by the lender at time \( T \), \( V_{\text{Lender}}(\sigma, \bar{D}, T) \) is obtained as the project’s asset value \( S(\sigma, T) \) minus the profit of the borrower, in other words,

\[
V_{\text{Lender}}(\sigma, \bar{D}, T) = S(\sigma, T) - v(\sigma, \bar{D}, T)
\]

\[
= \begin{cases} \min(S_t) > L & \text{max}[S_T - \bar{D}, 0] \min(S_t) > L \\ L & \text{otherwise} \\ 0 & \text{otherwise} \end{cases}
\]

\[
\vspace{0.5cm}
= \begin{cases} \min(S_T, \bar{D}) & \min(S_t) > L \\ L & \text{otherwise} \end{cases}
\]

At this time, in the case of

\[
\min(S_t) > L,
\]

when the assets of the project are available to repay the loan \( (S_T \geq \bar{D}) \), the lender can collect the whole amount of the promised loan. On the other hand, when \( S_T < \bar{D} \), default occurs, and the lender receives the asset value. In the case that the asset value falls below \( L \) only once before time \( T \), the lender perceives the situation, the project is immediately suspended, the contract between the lender and borrower is terminated, and the lender collects the asset, which has a value \( L \). According to Miyake and Inoue (2009), the project’s asset value \( S(\sigma, T) \) at maturity time \( T \) can be expressed in terms of a boundary \( L \) as follows:

\[
S(\sigma, T) = \begin{cases} S_T & \min(S_t) > L \\ L & \text{otherwise} \end{cases}
\]

\[
\vspace{0.5cm}
= \begin{cases} S_T & \min(S_t) > L \\ 0 & \text{otherwise} \end{cases} + \begin{cases} L & \min(S_t) > L \\ 0 & \text{otherwise} \end{cases}
\]

\[
= S_T e^{r(T-t)} \Phi(d_3) - \left( \frac{L}{S_t} \right)^{\frac{\sigma^2 t}{2}} \left( \frac{L^2}{S_t} e^{r(T-t)} N(d_4) + L - L \Phi(d_3 - \sigma \sqrt{T-t}) - \left( \frac{L}{S_t} \right)^{\frac{\sigma^2 t}{2}} L \Phi(d_4 - \sigma \sqrt{T-t}) \right).
\]
\[ d_3 = \frac{\ln(S_t / L) + (r + \sigma^2 / 2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_4 = \frac{\ln(L / S_t) + (r + \sigma^2 / 2)(T-t)}{\sigma \sqrt{T-t}}. \]

The payoff structures of (i) and (ii) consist of an asset digital option and a cash digital option involving a knock-out condition with a lower bound \( L \). Hence, the pricing for the values of (i) and (ii) is considered as follows.

Payoff of (i) = \( C_1 - C_2 \), where
\[ C_1 : \text{Payoff of } S_T \text{ obtained when the project’s asset value rises above the default boundary at maturity time } T. \]
\[ C_2 : \text{Payoff of } S_T \text{ obtained when the project’s asset value falls below the default boundary during the period and then recovers to above the default boundary at maturity time } T. \]

Value of (ii) = \( C_3 - C_4 \), where
\[ C_3 : \text{Payoff of } L \text{ obtained when the project’s asset value rises above the default boundary at maturity time } T. \]
\[ C_4 : \text{Payoff of } L \text{ obtained when the project’s asset value falls below the default boundary sometime during the period and then recovers to above the default boundary at maturity time } T. \]

**Remark 1:** The digital option was introduced by Hakansson (1976) and eventually inspired the establishment of exchange traded funds about a decade after the paper was published. Then, Rubinstein and Reiner (1991) introduced the cash digital option and developed a closed-form analytical formula within the B-S framework. Cox and Rubinstein (1985) obtained a formula for evaluating an asset digital option.

**Remark 2:** To avoid the risk-shifting incentive problem, any profit-sharing rule between the lender and borrower can be characterized by a fixed payment and a certain number of put and call options (Ziegler (2004) and Inoue et al. (2008)). Actually, at any point in time, there exist an infinite number of profit-sharing rules that avoid risk-shifting, although only one of these rules is feasible.

The payoff of the lender \( V_{\text{lender}}(\sigma, D, T) \) in (6) is a decreasing function of \( \sigma \). Therefore, the optimal risk \( \sigma^*_{\text{lender}} \) that maximizes the possible loan collected is obtained when \( \sigma \to 0 \), and to find an approximate value of \( \sigma^*_{\text{lender}} \), for example, under the assumption in the beginning of section 2, we simply let
\[ V_{\text{lender}}(\sigma, D, T)|_{\max \sigma \geq \sigma^*_{\text{lender}}} = I. \] (8)

In this case, we seek values less than the maximum value of \( \sigma \) so that the loan that can be collected is equal to the investment capital \( I = D \). In other words, we aim to find the value \( \sigma \) such a way that the above relation holds. Summarizing the payoff for the lender and borrower, the following table is obtained.

**Table 1. Payoff of lender and borrower.**

<table>
<thead>
<tr>
<th>Payoff of borrower</th>
<th>( v(\sigma, D, T) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff of lender</td>
<td>( S(\sigma, T) - v(\sigma, D, T) )</td>
</tr>
<tr>
<td>Total</td>
<td>( S(\sigma, T) )</td>
</tr>
</tbody>
</table>

**Safe and risky projects**

We assume two different types of project, a safe project \( S \) and a risky project \( R \), for which the risks are \( \sigma_S \) and \( \sigma_R \), respectively. Here, \( \sigma_S \) is assumed equal to the risk \( \sigma^*_{\text{lender}} \) obtained by (8) that maximizes the lender’s payoff, and \( \sigma_R \) is assumed equal to the risk \( \sigma^*_{\text{borrower}} \) obtained by (5) that maximizes the borrower’s payoff. Therefore, the relation of the risk of projects \( S \) and \( R \) is \( \sigma_R > \sigma_S \). When a borrower carries out project \( S \) under such an assumption, since the lender can collect the whole of the promised loan, it becomes the optimal policy for the lender to carry out project \( S \).

\[
\begin{align*}
\text{Project} S &\quad \text{Risk} : \sigma_S = \sigma^*_{\text{lender}} \\
\text{Project} R &\quad \text{Risk} : \sigma_R = \sigma^*_{\text{borrower}}
\end{align*}
\]
On the other hand, from the borrower’s standpoint, the profit $V_{\text{Borrower}}(\sigma_S, D, T)$ of the borrower when selecting project $S$ is lower than the profit $V_{\text{Borrower}}(\sigma_R, D, T)$ when selecting project $R$. That is, 

$V_{\text{Borrower}}(\sigma_R, D, T) > V_{\text{Borrower}}(\sigma_S, D, T)$.

Therefore, the borrower has an incentive to select the risky project $R$ to maximize his profit, contrary to the lender’s intention. In particular, if the borrower is a levered firm, the incentive to select the risky project $R$ becomes greater. Thus, the profit of the lender may be adversely affected and the risk-shifting incentive problem comes into existence between the lender and the borrower. However, if the lender is sensible and predicts the risk incentive of a borrower carrying out project $R$ in advance, and insists on a new loan $D^*$ satisfying

$$V_{\text{Lender}}(\sigma_R, D, T) \big|_{D=D^*} = I.$$ (9)

Thus, since $V_{\text{Lender}}(\sigma_R, D^*, T) = I$ in (9) is $S(\sigma_R, T) - V_{\text{Borrower}}(\sigma_R, D^*, T) = I$, the profit of the borrower becomes

$$V_{\text{Borrower}}(\sigma_R, D^*, T) = S(\sigma_R, T) - I.$$ (10)

Therefore, the difference between the profit of the borrower when selecting project $S$ and that of the borrower when selecting project $R$ under the new loan $D^*$ stipulated by the lender becomes

$$V_{\text{Borrower}}(\sigma_S, D, T) - V_{\text{Borrower}}(\sigma_R, D^*, T) = S(\sigma_S, T) - S(\sigma_R, T) + I.$$ (11)

Moreover, since $V_{\text{Lender}}(\sigma_S, D, T) = I$, (11) becomes

$$V_{\text{Borrower}}(\sigma_S, D, T) - V_{\text{Borrower}}(\sigma_R, D^*, T) = S(\sigma_S, T) - S(\sigma_R, T).$$ (12)

The project’s value $S(\sigma, T)$ in (7) involves the default boundary; thus, $S(\sigma, T)$ is a decreasing function of $\sigma$. The relation of the project’s asset value for the borrower carrying out projects $S$ and $R$ is $S(\sigma_S, T) > S(\sigma_R, T)$. Therefore, the right hand of (12) is positive, and the difference is an agency cost that the borrower must bear to pay. This additional cost increases with the difference between $S(\sigma_S, T)$ and $S(\sigma_R, T)$.

**Example 1.** Let $S_0 = 200$, $I = D = 200$, $r_d = 0.1$, $r = 0.2$, $T = 2$, $L = 160$. Then, for projects $S$ and $R$, $\sigma_S = 0.0193$ and $\sigma_R = 0.4657$ are obtained from (5) and (8). Figure 1 shows the transition of the borrower’s payoff obtained by (3) and the lender’s payoff obtained by (6) for different risks of the projects.

![Fig. 1. Payoff of lender and borrower for different risks of projects](image-url)
Figure 1 shows that as $\sigma$ increases to above 0.0193, the loan collected by the lender rapidly decreases and finally converges to the default boundary of 160. In contrast, the profit of the borrowing firm increases with increasing $\sigma$, reaches a maximum at $\sigma = 0.4657$, then it gradually decreases to a value of around 60. Thus, if the borrower chooses the safe project $S$ in accordance with the lender’s intention, the lender can collect the whole loan. However, if the borrower chooses the risky project $R$, the loan collected by the lender drops from 244.281 to 186.945 and default occurs, while the profit of the borrower increases from 54.0844 to 72.627.

<table>
<thead>
<tr>
<th>Table 2. Payoff of lender and borrower for projects $S$ and $R$.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Payoff of borrower</strong></td>
</tr>
<tr>
<td>Payoff of borrower</td>
</tr>
<tr>
<td>Payoff of lender</td>
</tr>
</tbody>
</table>

Here, if the lender is sensible to the situation and predicts the risk incentive of the borrowing firm in advance, and a new loan price of $D' = 573.36$ is required for the supply of investment capital $I = 200$ from (9). That is, the lender demands the expected rate of return of $200 \times \exp(2\times r_D) = 573.36$ so that $r_D = 0.526598$ for $T = 2$. In this case, the lender’s profit becomes 244.281. On the other hand, the payoff of the borrower becomes 15.2912 from (10). Therefore, the difference between the profit of the borrower when selecting project $S$ and that when selecting project $R$ under the loan $D'$ stipulated by the lender is 54.0844-15.2912 = 38.7932. This difference is considered an agency cost that the borrower must bear to pay.

**Effect of a second additional asset used as security**

A security is an asset that the lender can seize if the borrower does not repay the loan. According to Chan and Kanatas (1985) and Ziegler (2004) such a security can usually be classified into two types. One type of security is an existing asset of the borrower that is pledged to a lender in the event of default. The project’s asset in which the borrower has invested is the security for the loan, and the lender can seize the project’s asset if the borrower cannot repay the loan. In the other type, the security is an additional asset (hereafter second security) that will not normally be available to the lender.

In this section we consider the effect of a second additional asset of security on the risk incentive for the lender and borrower. Suppose that the lender and borrower agree on a security of a fixed amount $x$ at loan maturity time, and that the structure of the contract is the same as that in the previous section. We consider the payoff and optimal risk for the lender and borrower at the time of loan maturity.

**Borrower’s payoff and optimal risk**

When the lender and borrower agree on a security of a fixed amount $x$ at loan maturity time $T$, the profit of the borrower at time $T$ can be expressed as

$$V_{\text{Borrower}}(\sigma, D, T, x) = \begin{cases} \max[x + S_T - D, 0] & \min(S_D) > L \\ \max[x + L - D, 0] & \text{otherwise} \end{cases}.$$  \hspace{1cm} (13)

Next, we examine the profit of the borrower using (13) separately for two cases: (i) $D = x$, (ii) $x < D$.

(i) For $D = x$, namely, when the lender’s loan is fully collateralized by the second additional asset $x$, (13) becomes the same results as in (7), i.e.

$$V_{\text{Borrower}}(\sigma, D, T, x) = S(\sigma, T).$$  \hspace{1cm} (14)

(ii) For $x < D$, (13) can be examined in two cases: (a) $x + L = D$ (b) $x + L < D$.

(a) For $x + L = D$, namely, when the lender’s loan is completely guaranteed by the sum of $x$ and $L$, (13) becomes
\[
V_{\text{Borrower}}(\sigma, D, T, x) = \begin{cases} 
  x + S_T - D & \min_{t \in [0,T]} (S_t) > L \\
  0 & \text{otherwise} 
\end{cases} = x + S(\sigma, T) - D. 
\] (15)

(b) For \( x + L < D \), (13) becomes
\[
V_{\text{Borrower}}(\sigma, D, T, x) = \max[x + S_T - D, 0] \min_{t \in [0,T]} (S_t) > L = v(\sigma, D - x, T). 
\] (16)

\( v(\sigma, D - x, T) \) can be considered as the down-out-call-type knock-out option with \( D \) in (3) being replaced by \( D - x \). Therefore, the evaluation formula becomes
\[
v(\sigma, D - x, T) = S_t e^{r(T-t)} \Phi(d_5) - (D - x) \Phi(d_5 - \sigma \sqrt{T-t}) - S_t \left( \frac{S_t}{L} \right)^{-\frac{1}{2\sigma^2}} 
\]
\[
\times e^{r(T-t)} \Phi(d_6) + \left( \frac{S_t}{L} \right)^{-\frac{1}{2\sigma^2}} (D - x) \Phi(d_6 - \sigma \sqrt{T-t}), 
\] (17)

where \( d_5 = \frac{\ln(S_t, l(D - x)) + (r + \sigma^2 / 2)(T-t)}{\sigma \sqrt{T-t}} \), \( d_6 = \frac{\ln(L^2 / S_t, (D - x)) + (r + \sigma^2 / 2)(T-t)}{\sigma \sqrt{T-t}} \).

The derivative of the profit of the borrower \( V_{\text{Borrower}}(\sigma, D, T, x) \) with respect to \( \sigma \) is
\[
\frac{\partial V_{\text{Borrower}}}{\partial \sigma} = S_t e^{r(T-t)} \frac{\sqrt{T-t}}{2\pi} \left( \exp \left( -\frac{d_5^2}{2} \right) - \left( \frac{S_t}{L} \right)^{-\frac{1}{2\sigma^2}} \exp \left( -\frac{d_6^2}{2} \right) \right) 
\]
\[
- \frac{4r}{\sigma^2} \ln \left( \frac{S_t}{L} \right) \left( \frac{S_t}{L} \right)^{-\frac{1}{2\sigma^2}} 
\]
\[
\times S_t e^{r(T-t)} \Phi(d_5) - (D - x) \left( \frac{S_t}{L} \right)^{-\frac{1}{2\sigma^2}} \Phi(d_6 - \sigma \sqrt{T-t}). 
\] (18)

Thus, as similarly in (5), the optimal risk \( \sigma^*_{\text{Borrower}} \) that maximizes the borrower’s profit is found at the default boundary \( L \) where
\[
\left. \frac{\partial V_{\text{Borrower}}}{\partial \sigma} \right|_{\sigma = \sigma^*_{\text{Borrower}}} = 0. 
\] (18a)

Then, (5) can be solved numerically by showing some example because of the absence of a known closed-form solution.

**Lender’s payoff and optimal risk**

The possible loan collected by the lender at time \( T \), \( V_{\text{Lender}}(\sigma, D, T, x) \), is expressed as the total asset value including the security minus the borrower’s profit given by (13), in other words,
\[
V_{\text{Lender}}(\sigma, D, T, x) = x + S(\sigma, T) - V_{\text{Borrower}}(\sigma, D, T, x) 
\]
\[
= \begin{cases} 
  \min[x + S_T, D] & \min_{t \in [0,T]} (S_t) > L \\
  \min[x + L, D] & \text{otherwise} 
\end{cases}. 
\] (19)

We examine the value of the lender’s loan given by (19) for two cases: (i) \( D = x \), (ii) \( x < D \).

(i) For \( D = x \), namely, when the loan is fully collateralized, (19) becomes
\[
V_{\text{Lender}}(\sigma, D, T, x) = D. 
\] (20)

In this state, since the claim to the lender’s loan is completely guaranteed by the security, default cannot arise.
(ii) For \( x < \bar{D} \), (19) holds and two different cases are considered: (a) \( x + L = \bar{D} \), (b) \( x + L < \bar{D} \).

(a) For \( x + L = \bar{D} \), (19) becomes

\[
V_{\text{Lender}}(\sigma, \bar{D}, T, x) = \bar{D}. \tag{21}
\]

(b) For \( x + L < \bar{D} \), (19) becomes

\[
V_{\text{Lender}}(\sigma, \bar{D}, T, x) = \begin{cases} 
\min[x + S_T, \bar{D}] & \min(S_T) > L \\
x + L & \text{otherwise}
\end{cases},
\]

which leads to

\[
V_{\text{Lender}}(\sigma, \bar{D}, T, x) = \begin{cases} 
x + S_T - \max[x + S_T - \bar{D}, 0] & \min(S_T) > L \\
x + L & \text{otherwise}
\end{cases}
\]

\[
= x + S(\sigma, T) - v(\sigma, \bar{D} - x, T) \tag{23}
\]

At this time, \( v(\sigma, \bar{D} - x, T) \) is the borrower’s profit given by (16). Summarizing the payoff for the lender and borrower, Table 3 is obtained.

### Table 3. Payoff for the lender and borrower.

<table>
<thead>
<tr>
<th>( x = \bar{D} )</th>
<th>( x &lt; \bar{D} )</th>
<th>( x + L = \bar{D} )</th>
<th>( x + L &lt; \bar{D} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff of borrower</td>
<td>( S(\sigma, T) )</td>
<td>( x + S(\sigma, T) - \bar{D} )</td>
<td>( v(\sigma, \bar{D} - x, T) )</td>
</tr>
<tr>
<td>Payoff of lender</td>
<td>( \bar{D} )</td>
<td>( \bar{D} )</td>
<td>( x + S(\sigma, T) - v(\sigma, \bar{D} - x, T) )</td>
</tr>
<tr>
<td>Total</td>
<td>( x + S(\sigma, T) )</td>
<td>( x + S(\sigma, T) )</td>
<td>( x + S(\sigma, T) )</td>
</tr>
</tbody>
</table>

Only when the amount of security is \( x = \bar{D} \) and \( x + L = \bar{D} \), in other words, when the lender’s loan is completely guaranteed, there is no risk-shifting incentive problem for the lender and borrower. The reason for this is that the lender’s payoff does not depend on the project’s risk. Therefore, even if the borrower chooses a high-risk project, the lender’s profit does not decrease. On the other hand, when the amount of security is \( x + L < \bar{D} \), the risk-shifting incentive problem comes into existence between the lender and borrower. Here, the optimal risk \( \sigma^{**}_{\text{Borrower}} \) that maximizes the profit of the borrower given in (16) is obtained with (18)

\[
\frac{\partial V_{\text{Borrower}}(\sigma, \bar{D}, T, x)}{\partial \sigma} \bigg|_{\sigma = \sigma^{**}_{\text{Borrower}}} = 0. \tag{24}
\]

On the other hand, the project risk at which the possible loan collected by the lender given by (23) is maximized is

\[
V_{\text{Lender}}(\sigma, \bar{D}, T, x) \bigg|_{\sigma = \sigma^{**}_{\text{Lender}}} = I. \tag{25}
\]

Here, we assume two different types of project, a safe project \( S \) and a risky project \( R \), for which the risks are \( \sigma_S \) and \( \sigma_R \), respectively. Here, \( \sigma_S \) is equal to the risk \( \sigma^{**}_{\text{Lender}} \) obtained by (25) and \( \sigma_R \) is equal to the risk \( \sigma^{**}_{\text{Borrower}} \) obtained by (24). Therefore, from the lender’s standpoint, it becomes the optimal policy to carry out Project \( S \).

\[
\begin{cases} 
\text{Project} \overset{\text{Risk}}{S} & \text{Risk:} \sigma_S = \sigma^{**}_{\text{Lender}} \\
\text{Project} \overset{\text{Risk}}{R} & \text{Risk:} \sigma_R = \sigma^{**}_{\text{Borrower}}
\end{cases}
\]

On the other hand, from the borrower’s standpoint, the profit of the borrower when carrying out project \( S \) is lower than that of the borrower when carrying out project \( R \). That is,

\[
V_{\text{Borrower}}(\sigma_{R}, \bar{D}, T, x) > V_{\text{Borrower}}(\sigma_{S}, \bar{D}, T, x). 
\]
Therefore, it becomes the borrower’s optimal policy to carry out Project R. Here, the lender predicts the borrower’s risk incentive in advance and insists on a new loan price \( D^{**} \) satisfying
\[
V_{\text{Lender}}(\sigma_R, \bar{D}, T, x) \bigg|_{D = D^{**}} = I. \tag{26}
\]

Since \( V_{\text{Lender}}(\sigma_R, D^{**}, T, x) = I \) in (26) is \( x + S(\sigma_R, T) - V_{\text{Borrower}}(\sigma_R, D^{**}, T, x) = I \), the borrower’s profit becomes
\[
V_{\text{Borrower}}(\sigma_R, D^{**}, T, x) = x + S(\sigma_R, T) - I. \tag{27}
\]

Therefore, the difference between the profit of the borrower when selecting project S and that of the borrower when selecting project R under the new loan \( D^{**} \) becomes
\[
V_{\text{Borrower}}(\sigma_s, \bar{D}, T, x) - V_{\text{Borrower}}(\sigma_R, D^{**}, T, x) = x + S(\sigma_s, T) - V_{\text{Lender}}(\sigma_s, \bar{D}, T, x)
- x - S(\sigma_R, T) + I. \tag{28}
\]

Since \( S(\sigma_s) > S(\sigma_R) \) and \( V_{\text{Lender}}(\sigma_s, \bar{D}, T, x) = I \), (28) becomes
\[
V_{\text{Borrower}}(\sigma_s, \bar{D}, T, x) - V_{\text{Borrower}}(\sigma_R, D^{**}, T, x) = S(\sigma_s, T) - S(\sigma_R, T) > 0. \tag{29}
\]

The difference showed by (28) is positive, indicating the existence of an agency cost that the borrower pays. Next, we illustrate the effect of a second additional asset of security on the borrower’s agency cost in Example 2.

**Example 2.** Let \( S_0 = 200 \), \( I = D = 200 \), \( r_d = 0.1 \), \( r = 0.2 \), \( T = 2 \), \( L = 160 \), as in Example 1, and suppose that a lender and a borrower agree on a security of a fixed amount \( x = 20 \) at loan maturity time. Then, for projects S and R, \( \sigma_s = 0.0263 \) and \( \sigma_R = 0.3042 \) are obtained from (24) and (25). Figure 2 shows the transition of the borrower’s payoff (16) and the lender’s payoff (23) for different risks of the projects.

![Fig. 2. Payoff of lender and borrower for different risks of projects in the case of \( x = 20 \)](image)
If the borrower chooses the safe project $S$ in accordance with the lender’s intention, the lender can collect the whole loan. However, if the borrower chooses the risky project $R$, the loan collected by the lender drops from 244.281 to 217.9 and default occurs, while the profit of the borrower increases from 74.0844 to 80.7258. Here, suppose that the lender is sensible and perceives the risk incentive of the borrower in advance, and a new loan price of $D'' = 345.34$ is required for the supply of investment capital $I = 100$ from (26). That is, the lender demands the expected rate of return of $(200 \times \exp(2 \times r_d) = 345.34 \rightarrow r_d = 0.2731$ for $T = 2$. When the borrower carries out project $R$ under the new loan $D''$, the lender’s payoff becomes 244.281. On the other hand, the payoff of the borrower is 54.3451 from (27). Therefore, the borrower’s agency cost becomes 74.0844-54.3451=19.7393. When we compare the agency cost in the case of $x = 0$ shown in Example 1 with that in the case of $x = 20$, the borrower’s agency cost is decreased by 38.7932-19.7393=19.0539, where 38.7932 is the agency cost for $x=0$ shown in Example 1. Next, Table 5 shows the borrower’s agency cost for $x = 0, 10, 20, 30, 50$.

### Table 4. Payoff of lender and borrower for projects $S$ and $R$ in the case of $x = 20$.  

<table>
<thead>
<tr>
<th>Project $S$</th>
<th>Project $R$ (Loan $D = 244.281$)</th>
<th>Project $R$ (New loan $D' = 345.34$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff of borrower</td>
<td>74.0844</td>
<td>80.7258</td>
</tr>
<tr>
<td>Payoff of lender</td>
<td>244.281</td>
<td>217.9</td>
</tr>
</tbody>
</table>

### Table 5. Agency cost for different amounts of security. 

<table>
<thead>
<tr>
<th>$\sigma_R$</th>
<th>$\sigma_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>0.4657</td>
</tr>
<tr>
<td>$x = 10$</td>
<td>0.3788</td>
</tr>
<tr>
<td>$x = 20$</td>
<td>0.3042</td>
</tr>
<tr>
<td>$x = 30$</td>
<td>0.2407</td>
</tr>
<tr>
<td>$x = 50$</td>
<td>0.1395</td>
</tr>
</tbody>
</table>

(\(\sigma_R\) : borrower’s optimal risk obtained by (24), \(\sigma_S\) : lender’s optimal risk obtained by (25))

The agency cost that the borrower is supposed to pay decreases with increasing of the second security $x$. Finally, in the case of $x = 84.281$ (244.281-160), in other word, when the lender’s loan is completely guaranteed by the sum of $x$ and $L$, the borrower’s agency cost becomes 0. Thus, the existence of a security reduces a borrower’s agency cost. The reason for this is that the security mitigates the risk-shifting incentive problem for the lender and the borrower. For example, the borrower’s optimal risk shifts to that of the safer project with increasing $x$ and finally becomes 0 at $x=84.281$. On the other hand, the lender has less interest in the risk incentive of the borrowing firm with increasing $x$, and finally the lender’s risk incentive disappears completely at $x=84.281$. In this respect, a security can be understood as an effective way of reducing the incentive problem and the borrower’s agency cost.

### Concluding remarks

In this paper, we studied not only the borrower’s risk incentive but also the lender’s risk incentive and their relationship. We focus on an effective financing strategy of a lender for which the lender’s expected profit is valued by taking borrower’s risk incentive into account. In order to proceed with the analysis we used a knock-out option for the borrower’s payoff so that the optimal volatility level of projects becomes finite. Thus, the standard B-S evaluation model is not sufficient when volatility is sensitive to asset values and borrowers are supposed to always select infinite-
volatility projects. In general, the possibility of moral hazard raised by borrower’s firm is considered as one of several causes which the borrower himself may make agency cost high. Through our numerical simulations, it is observed that when the lender perceives the possibility of risk incentive of the borrower he may impose to require an additional amount to be paid, so that the borrower will refrain from investing in the risky project, instead, he would use a second additional asset as security to be able to maintain or recover his trust or credibility. This process shows to mitigate the risk incentive problems.

In summarizing, we discussed and examined the effect of a second additional asset as security on the risk incentive for the lender and the borrower. The existence of this security has the following advantages for the lender and the borrower. From the lender’s standpoint, the security grants the lender a claim on an additional asset in the case of bankruptcy, thus allowing him to recover more of the capital. Moreover, in the extreme case that the lender’s loan is completely guaranteed by the security, the borrower’s risk-shifting incentive disappears and the possibility that default is caused will become zero. On the other hand, from the borrower’s standpoint, the security reduces the difference between the risks of different projects, as desired by the lender and borrower, thus bringing a reduction in the agency cost paid by the borrower and recovering borrower’s creditability. Through option pricing theory, it is found that a security can be an effective way of reducing the incentive problem and the borrower’s agency cost, so that the risk incentive problems are mitigated.

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References