Credibility measures in portfolio analysis: From possibilistic to probabilistic models

Irina Georgescu 1 and Jani Kinnunen 2,*

1 Academy of Economic Studies, Department of Economic Cybernetics, Bucharest, Romania
2 Institute for Advanced Management Systems Research, Åbo Akademi University, Turku, Finland

Abstract. This paper treats risk based on the notions of credibility measure and credibility expected value. Firstly, the paper derives and discusses the credibility expected value. Secondly, the paper presents a new method of analysis of possibilistic portfolios. The new step is a construction by which with a possibilistic portfolio one associates a probabilistic portfolio. The problem solving of possibilistic portfolio gets down to the problem solving of associated probabilistic portfolio. For the latter one a variety of solving methods exist, from which we can choose the most appropriate one for the initial problem. Risk evaluation in the context of probabilistic portfolio leads to an understanding of risk for the possibilistic portfolio. The paper presents an application case of a venture capitalist firm, which needs to solve the shares of its budget to be invested to start-up companies.

Keywords: credibility measures; risk theory; probability measures; portfolio analysis

Received October 2010. Accepted April 2011

Introduction

Probabilistic risk is studied by probabilistic indicators such as expected value, variance, covariance, etc. In possibility theory, initiated by Zadeh (1965; 1978), the place of the probability measure is taken by the notions of possibility measure and necessity measure, and random variables are replaced by possibilistic distributions (Dubois and Prade, 1987; 1988). Possibilistic concepts of expected value, variance, covariance, etc. should be replaced by appropriate ones in possibility theory. Fuzzy numbers constitute the most studied class of possibility distributions and for them this desideratum has been realized to a great extent in papers of Carlsson and Fullér (2001; 2002), Fullér (2000), Fullér and Majlender (2002), and Liu (2007a). The concepts of expected value, variance and covariance introduced in these papers have good mathematical properties, which allow their efficient use in applications (see Carlsson and Fullér (2002), Carlsson, Fullér, and Majlender (2002), Fullér (2000), and Majlender (2004)). The notion of credibility measure introduced in Liu and Liu (2002) leads to a new way of defining and studying possibilistic indicators. Unlike possibility measure, credibility measure is a self–dual measure, which confers it remarkable mathematical properties (see Liu, 2007a; 2007b).

The analysis of the problem of portfolio selection by mean-variance models of Markowitz (1952) is one of the main themes of financial mathematics. From models like Markowitz’s which tackle the behaviour of investment under uncertainty we focused over those with the following optimization principle: the setting of the return to a
certain level and the minimization of the investment risk. The principle of optimization is formulated in probabilistic context: the expected value of return should reach a certain level $\lambda$ and the variance of the portfolio is minimized. Possibilistic mean-variance models replace probabilistic indicators by possibilistic indicators: a possibilistic expected value of return will reach the level $\lambda$ and a possibilistic variance will be minimized. Among these possibilistic mean-variance models we find some based on credibility theory (Wang and Zhu 2002; Chen et al. 2006; Huang 2009; Huang 2011). In these papers credibility expected value and credibility variance appear as indicators.

This paper proposes a new model of portfolio analysis. It is based on a construction from (Liu and Liu 2002; Liu 2007a) by which a random variable $X$ is associated with a discrete fuzzy variable $\xi$. The main property of this construction is that the credibility expected value of $\xi$ equals the probabilistic expected value of $X$. The main contribution of the paper is based on the idea: with a possibilistic portfolio we canonically associate a probabilistic portfolio. The possibilistic portfolio problem is replaced by a probabilistic portfolio problem. An optimal solution of the latter is an optimal solution of the former. There are several ways of solving the probabilistic portfolio problem. In our case we can choose the most adequate for solving the initial possibilistic portfolio problem.

In Section 2 we recall the notions of fuzzy measure, possibility measure and necessity measure and the relationship between the last two of them. By using Liu (2007a), we introduce axiomatically the concept of credibility measure. To each possibility measure a credibility measure is canonically associated. Section 3 deals with the credibility measures associated with possibility distributions. Starting with a possibility distribution $\mu: \mathbb{R} \rightarrow [0,1]$ we can take the possibility measure $\text{Pos}_\mu$, associated with $\mu$ and further, we can consider the credibility measure $\text{Cr}_\mu$ defined by $\text{Pos}_\mu$. All credibility measures of this paper will have the form $\text{Cr}_\mu$. The credibility expected value $Q(\xi)$ associated with a fuzzy variable $\xi$ is defined in Section 4. It is analyzed especially the case of a fuzzy variable $\xi$ with a finite number of values (= simple possibility distribution). In this situation, to the fuzzy variable $\xi$ we can associate a random variable $X_\xi$ whose expected value is equal with $Q(\xi)$. This makes a possibilistic model described by a simple fuzzy variable be consistent with a probabilistic model. Section 5 discusses a notion of possibilistic portfolio based on the construction presented in Section 4. To a possibilistic portfolio one associates a probabilistic portfolio such that the risk problematics for the first type of portfolio is reduced to the second one. Section 6 presents a calculation example of a venture capitalist investing to two start-up companies, and Section 7 concludes the paper.

**Credibility measures**

Credibility measure is a particular fuzzy measure; it was defined in Liu and Liu (2002) by means of the notions of possibility measure and necessity measure. In Liu (2007a) the notion of credibility measure is introduced by axioms without referring to possibility measure and necessity measure (see Definition 1 below). The notion of credibility allowed the development of a new field, called credibility theory, with consistent mathematical results and with significant applications (see the references of Liu (2007a)). In this section the notions of possibility measure, necessity measure, credibility measure and the relation between them are recalled.

Let $\Omega$ be a non–empty set and $P(\Omega)$ the power set of $\Omega$. The elements of $\Omega$ are interpreted as states, and the sets of $P(\Omega)$ as events. A fuzzy measure on $\Omega$ is a function $m: P(\Omega) \rightarrow [0,1]$ such that the following conditions hold:

(M1) $m(\emptyset) = 0; m(\Omega) = 1$;

(M2) If $D_1, D_2 \in P(\Omega)$ then $D_1 \subseteq D_2$ implies $m(D_1) \leq m(D_2)$.

The notion of fuzzy measure is more general than probability. Axiom (M2) is a condition of compatibility of gradation of measures of events with respect to their inclusion. A possibility measure on $\Omega$ is a function $\Pi: P(\Omega) \rightarrow [0,1]$ such that the following conditions are verified:

(P1) $\Pi(\emptyset) = 0; \Pi(\Omega) = 1$;

(P2) For any family $\{D_i\}_{i \in I}$ of subsets of $\Omega$, $\Pi(\bigcup_{i \in I} D_i) = \sup_{i \in I} \Pi(D_i)$.

A necessity measure on $\Omega$ is a function $N: P(\Omega) \rightarrow [0,1]$ such that the following conditions are verified:
If \( D \) is an event then \( \Pi(D) \) is the possibility of occurrence of \( D \) and \( N(D) \) is the necessity of occurrence of \( D \).

**Proposition 1** (Dubois and Prade, 1987; 1988) Any possibility measure (resp. any necessity measure) on \( \Omega \) is a fuzzy measure. For any \( D \in P(\Omega) \) we denote \( D^c = \Omega - D \).

**Proposition 2** (Dubois and Prade, 1987; 1988)

If \( \Pi \) is a possibility measure on \( \Omega \) then the function \( \text{Nec}_\Pi : P(\Omega) \to [0,1] \) defined by \( \text{Nec}_\Pi(D) = 1 - \Pi(D^c) \) for any \( D \in P(\Omega) \) is a necessity measure;

If \( N \) is a necessity measure on \( \Omega \) then the function \( \text{Pos}_N : P(\Omega) \to [0,1] \) defined by \( \text{Pos}_N(D) = 1 - N(D^c) \) for any \( D \in P(\Omega) \) is a possibility measure;

The functions \( \Pi \to \text{Nec}_\Pi \) and \( N \to \text{Pos}_N \) are inverse to one another and establish a bijective correspondence between the measures of possibility on \( \Omega \) and the measures of necessity on \( \Omega \).

The notions of possibility measure and necessity essentially differ from probability: in the definition of the latter a property of metric nature appears (the probability of “sum” of two incompatible events is the sum of their probabilities), while conditions (P2) and (N2) express the preservation of infinite logical operations. A fuzzy measure \( m \) on \( \Omega \) is self–dual if \( 1)(\sup_i I_i D) = \Pi(D) + \Pi(D^c) \) for any \( D \in P(\Omega) \). The measures of possibility and necessity are not self–dual. The property of a fuzzy measure of being self–dual has important consequences. It appears in the definition of the concept of credibility measure, introduced by Liu and Liu (2002).

**Definition 1** A credibility measure on \( \Omega \) is a function \( \text{Cr} : P(\Omega) \to [0,1] \) such that \( \text{Cr}(\Omega) = 1 \);

If \( A,B \in P(\Omega) \) then \( A \subseteq B \) implies \( \text{Cr}(A) \leq \text{Cr}(B) \);

For any \( A \in P(\Omega), \text{Cr}(A) + \text{Cr}(A^c) = 1 \);

For any family \( \{A_i\}_{i \in I} \) of subsets of \( \Omega \) with the property that \( \sup_i \text{Cr}(A_i) \leq \frac{1}{2} \) we have \( \text{Cr}(\bigcup_i A_i) = \sup_i \text{Cr}(A_i) \).

**Remark 1** According to definition (1) \( \text{Cr}(\emptyset) = 0 \).

The elements of \( P(\Omega) \) are interpreted as events. If \( A \) is an event then \( \text{Cr}(A) \) represents the credibility with which \( A \) occurs. The following result shows how a possibility measure can lead to a credibility measure.

**Proposition 3** Let \( \Pi \) be a possibility measure on \( \Omega \). We define the function \( \text{Cr} : P(\Omega) \to [0,1] \) by \( \text{Cr}(A) = \frac{1}{2} [\Pi(A) + 1 - \Pi(A^c)] \) for any \( A \in P(\Omega) \). Then \( \text{Cr} \) is a credibility measure on \( \Omega \).

One can prove that any credibility measure can be expressed in form of Proposition 3 (see Liu 2007b). This result shows that the credibility of an event is the arithmetic mean of its possibility and necessity, which is more suggestive than its axiomatic definition.

**Possibility distributions**

This section presents possibilistic distributions related to probability measure and credibility measure. In this section we consider only possibility measures and credibility measures on the set \( \mathbb{R} \) of real numbers. A possibility distribution is a function \( \mu : \mathbb{R} \to [0,1] \) such that \( \sup_{x \in \mathbb{R}} \mu(x) = 1 \); \( \mu \) is normalized if \( \mu(x_0) = 1 \) for some \( x_0 \in \mathbb{R} \). Let \( \Pi \) be a
possibility measure on \( R \). The function \( \mu : R \rightarrow [0,1] \) defined by \( \mu(x) = \Pi(\{x\}) \) for all \( x \in R \) is a possibility distribution. Let \( \mu \) be a possibility distribution. Let us consider the function \( Pos : P(R) \rightarrow [0,1] \) defined by

\[
Pos_\mu(D) = \sup_{x \in D} \mu(x) \text{ for any } D \in P(R). \tag{1}
\]

The following two propositions are known. They establish the relation between possibility, possibility distributions, and fuzzy variables.

**Proposition 4** \( Pos_\mu \) is a possibility measure on \( R \).

A fuzzy variable on \( R \) is an arbitrary function \( \xi : R \rightarrow R \). If \( B \) is a set of real numbers then we denote

\[
\{ \xi \in B \} = \{ x \in R \mid \xi(x) \in B \}.
\]

In particular, for any \( r \in R \) we can consider events as

\[
\{ \xi = r \} = \{ x \in R \mid \xi(x) = r \},
\]

\[
\{ \xi \geq r \} = \{ x \in R \mid \xi(x) \geq r \},
\]

etc.

We say that the possibility distribution \( \mu : R \rightarrow [0,1] \) is associated with the fuzzy variable \( \xi : R \rightarrow R \) if the following condition is fulfilled:

\[
Pos_\mu(\xi = x) = \mu(x) \text{ for any } x \in R. \tag{2}
\]

From (2) we notice that the possibility distribution \( \mu \) is uniquely determined by the fuzzy variable \( \xi \).

**Proposition 5** Let \( \xi \) be a fuzzy variable with the possibility distribution \( \mu \). For any \( A \subseteq R \) we have

\[
Pos_\mu(\xi \in A) = \sup_{x \in A} \mu(x) \tag{3}
\]

In particular, for any \( r \in R \),

\[
Pos_\mu(\xi \geq r) = \sup_{x \geq r} \mu(x) \tag{4}
\]

Let \( Cr_\mu : P(R) \rightarrow [0,1] \) be the credibility measure associated with \( Pos_\mu \) (according to Proposition 3). Then according to Proposition 3 the credibility of any event \( A \subseteq R \) is expressed by:

\[
Cr_\mu(A) = \frac{1}{2} [Pos_\mu(A) + 1 - Pos_\mu(A^c)] \tag{5}
\]

If \( \xi \) is a fuzzy variable with the possibility distribution \( \mu \), then for any \( A \subseteq R \) the credibility of the event \( \{ \xi \in A \} \) is computed by the formula:

\[
Cr_\mu(\xi \in A) = \frac{1}{2} [Pos_\mu(\xi \in A) + 1 - Pos_\mu(\xi \in A^c)] \tag{6}
\]

In particular, for any \( r \in R \) the credibility that \( \xi \) takes values smaller than \( r \) is

\[
Cr(\xi \leq r) = \frac{1}{2} [\sup_{x \geq r} \mu(x) + 1 - \sup_{x \leq r} \mu(x)] \tag{7}
\]

There are several experiments in which the set of outcomes can be expressed as a finite or infinite sequence. Then we deal with discrete random variables (in the probabilistic case) and discrete fuzzy variables (in the possibilistic case). A fuzzy variable \( \xi : R \rightarrow R \) is called discrete if the set \( \xi(R) = \{ \xi(x) \mid x \in R \} \) of its values is at most countable. If \( \xi(R) \) is finite then \( \xi \) is called ‘simple’. Let \( \xi \) be a discrete fuzzy variable with \( \xi(R) = \{ x_1, ..., x_n \} \), where \( x_1 < ... < x_n \). Let \( \mu : R \rightarrow [0,1] \) the possibility distribution associated with \( \xi \). If we denote \( \mu_i = \mu(x_i) \) for \( i = 1, ..., n \) then
\[ \text{Pos}_\mu(\xi = x_j) = \mu_j, \quad j = 1, \ldots, n \] (8)

One notices that for \( x \neq x_i \) for any \( i = 1, \ldots, n \) we have \( \text{Pos}_\mu(\xi = x) = 0 \). The above situation will be concentrated in the following table:

\[
\xi: \begin{bmatrix} x_1 & \cdots & x_n \\ \mu_1 & \cdots & \mu_n \end{bmatrix}
\] (9)

which signifies the fact that the fuzzy variable \( \xi \) takes the values \( x_1, \ldots, x_n \) with possibilities \( \mu_1, \ldots, \mu_n \). The equation (8) determines \( \xi \)'s possibilistic distribution. Therefore we call this table discrete possibilistic distribution.

**Credibility expected value**

In probability theory various indicators associated with random variables are studied: mean value, variance, covariance, etc. These indicators give information concentrated on the behaviour of random variables. When we pass from random variables (probability distributions) to possibility distributions one naturally asks the question of defining similar indicators. For fuzzy variables whose possibility distributions are fuzzy numbers, in Carlsson and Fullér (2001), Fullér and Majlender (2002), and Majlender (2004) etc. the possibilistic mean value, possibilistic dispersion, possibilistic covariance etc. are defined. These concepts have mathematical properties sufficiently rich which allow the development of the mathematical theory and its applicability.

On the other hand, their definitions are intrinsically connected with the form of the fuzzy number and cannot be extended to other classes of possibility distributions (e.g. discrete possibility distributions). Thus the open problem is to find indicators for larger classes of possibility distributions. This section is dedicated to a concept of mean value defined by Liu and Liu (2002) for any possibility distributions. It is different from the one from the case of fuzzy numbers (Carlsson and Fullér, 2001; Fullér and Majlender, 2002) and is based on the notion of credibility measure. Therefore, we will call it ‘credibility expected value’. For the case of a finite discrete possibility distribution, the credibility expected value has a very appropriate formula from the point of view of the calculation. At the same time in the discrete case an interesting phenomenon occurs: with a finite discrete fuzzy variable \( \xi \) a finite discrete random variable \( X_\xi \) is associated, whose mean value coincides with the credibility expected value of \( \xi \). This fact allows that some possibilistic decision problems to be converted into probabilistic decision problems. An optimal solution for the latter leads to an optimal solution for the former. Let \( \xi \) be a fuzzy variable with a normalized possibility distribution \( \mu \).

**Definition 2** (Liu and Liu, 2002; Liu, 2007a) The credibility expected value \( Q(\xi) \) of \( \xi \) is defined by

\[
Q(\xi) = \int_0^\infty \text{Cr}(\xi \geq r) \, dr - \int_{-\infty}^0 \text{Cr}(\xi \leq r) \, dr
\]

If the right hand side of Definition 2 has the form \( -\infty -\infty \), then \( Q(\xi) \) does not exist.

**Remark 2** (Liu and Liu, 2002) If \( \mu \) is the trapezoidal fuzzy number \( (a,b,\alpha,\beta) \) then

\[
Q(\xi) = \frac{1}{2}(a+b) + \frac{1}{4}(\beta - \alpha).
\]

By Liu and Liu (2002) and Liu (2007a) the possibilistic expected value \( E(\xi) \) in case when \( \mu = (a,b,\alpha,\beta) \) is

\[
E(\xi) = \frac{a+b}{2} + \frac{\beta - \alpha}{6}, \quad \text{and therefore} \quad Q(\xi) \neq E(\xi).
\]

Now let us assume that \( \xi \) is the normalized discrete possibility distribution:

\[
\mu: \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ \mu_1 & \mu_2 & \cdots & \mu_n \end{bmatrix}, \quad a_1 < a_2 \ldots < a_n
\] (10)
Recall that \( a_1, \ldots, a_n \) are the values of the fuzzy variable \( \xi \) and \( \mu_1, \ldots, \mu_n \) represent the possibilities with which \( \xi \) takes these values. Let \( \mu_0 = \mu_{n+1} = 0 \). For any \( i = 1, \ldots, n \), let us denote

\[
p_i = \frac{1}{2} \left( \bigvee_{j=1}^{i-1} \mu_j - \bigvee_{j=i+1}^{n} \mu_j \right) + \frac{1}{2} \left( \bigvee_{j=1}^{i} \mu_j - \bigvee_{j=i+1}^{n} \mu_j \right)
\]

(11)

**Proposition 6** (Huang, 2009; Liu and Liu, 2002) the numbers \( p_1, \ldots, p_n \) verify the following properties:

(i) \( p_i \geq 0 \) for any \( i = 1, \ldots, n \);

(ii) \( \sum_{i=1}^{n} p_i = \bigvee_{i=1}^{n} \mu_i = 1 \).

The proposition puts the stress on a remarkable fact: we can consider the discrete random variable

\[
X_\xi = \begin{pmatrix} a_1 & \ldots & a_n \\ p_1 & \ldots & p_n \end{pmatrix}
\]

(12)

in which \( p_1, \ldots, p_n \) are the probabilities with which \( X_\xi \) takes the values \( a_1, \ldots, a_n \).

**Proposition 7** (Liu and Liu, 2002)

\[
Q(\xi) = E(X_\mu) = \sum_{i=1}^{n} a_i p_i
\]

According to Propositions 6 and 7 with the fuzzy variable \( \xi \) we can associate a random variable \( X_\xi \), such that the credibility expected value \( Q(\xi) \) coincides with the probabilistic mean value \( E(X_\xi) \). In this way the situation of possibilistic uncertainty described by \( \xi \) and \( \mu \) can be probabilistically modelled by \( X_\xi \). In particular, some problems of possibilistic optimization can be treated as problems of probabilistic optimization. This idea will be discussed in the next section.

**Open problem 1** Let \( \xi \) be a fuzzy variable and \( Q(\xi) \) its credibility expected value. By Liu and Liu (2002) the credibilistic variance \( V[\xi] \) of \( \xi \) is defined by \( V[\xi] = Q(\xi - Q(\xi))^2 \). Assume that \( \xi \) is a discrete fuzzy variable defined by (12) and \( X_\xi \) is the associated discrete random variable (by Proposition 6). An open problem is whether there exists a relation between \( V[\xi] \) and the variance \( Var(X_\xi) \) of the random variable \( X_\xi \).

**Possibilistic portfolios**

A portfolio is a set of financial assets (money, bonds, etc.) and real goods (land, buildings, gold, etc.) ready to be bought. A portfolio is characterized by a return and by a risk related to bonds, which will take place in future. We consider two notions of portfolio: probabilistic portfolio and possibilistic portfolio. For the first one, the return is expressed in terms of probabilities and for the second one in terms of possibilities. For the case of probabilistic portfolio, the risk is given by the variance of a random variable. To define the risk of a possibilistic portfolio, we shall refer to a construction based on the contents of the previous section. With a possibilistic portfolio we canonically associate a probabilistic portfolio equivalent from the viewpoint of the return. The risk of the possibilistic portfolio will be the variance, which appears in case of the associated probabilistic portfolio. For the notions on probabilistic portfolio we refer to Stancu and Andrei (1997).

The possibilistic portfolio is made of \( m \) actions or (hereafter) assets \( A_1, \ldots, A_m \) and is defined by the following data: \( r_{ij} \) is the expected return for asset \( A_i \) at state \( j \); \( \mu_{ij} \) is the possibility of obtaining the expected return \( r_{ij} \) for asset \( A_i \) at state \( j \), where \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \). We denote by \( P^\leq r_{ij}, \mu_{ij}, i=1, \ldots, m, j=1, \ldots, n \) the possibilistic
portfolio defined above. The difference between the probabilistic portfolio and possibilistic portfolio is essential: in case of the first one, we consider the probability of obtaining a return, and in the second case the possibility of obtaining a return. The issue is that the probabilistic notions in case of the probabilistic portfolio (mean value, variance, covariance) to be replaced with possibilistic notions. We emphasize that in the first case we dealt with discrete random variables; in the second case we have discrete possibilistic distributions. We define the possibilistic return corresponding to asset \( A_i \) as the discrete possibilistic distribution:

\[
\xi_i = \begin{bmatrix} r_{i1} & r_{i2} & \cdots & r_{in} \\ \mu_{i1} & \mu_{i2} & \cdots & \mu_{in} \end{bmatrix}, \quad i = 1, \ldots, m .
\] (13)

The mean possibilistic return of asset \( A_i \) is defined as:

\[
\bar{\mu}_i = Q(\xi_i), \quad i = 1, \ldots, m
\] (14)

In (14) we used the credibility expected value introduced in Section 4. It would follow the introduction of possibilistic variance of an asset, but we do not have a satisfying notion of variance of a discrete possibility distribution. We consider the discrete random variable \( X_i \) associated with the fuzzy variable \( \xi_i \)

\[
X_i = \begin{bmatrix} r_{i1} & r_{i2} & \cdots & r_{in} \\ P_{i1} & P_{i2} & \cdots & P_{in} \end{bmatrix}, \quad i = 1, \ldots, m .
\]

where according to (12):

\[
p_{ik} = \frac{1}{2} \left[ \sqrt{k} \mu_{ij} - \sqrt{k-1} \mu_{ij} \right] + \frac{1}{2} \left[ \sqrt{n} \mu_{ij} - \sqrt{n+1} \mu_{ij} \right] \quad \text{for any } i = 1, \ldots, m \text{ and } k = 1, \ldots, n .
\] (15)

We recall that in (15) we take \( p_{i0} = p_{in+1} = 0, j = 1, \ldots, m \). The number \( p_{ik} \) of (15) represents the probability that the return of asset \( A_i \) is \( r_{ik} \). According to Proposition 7 the mean possibilistic return of asset \( A_i \) can be expressed by

\[
\bar{\mu}_i = Q(\xi_i) = E(X_i) = \sum_{k=1}^{n} r_{ik} p_{ik} \quad \text{for any } i = 1, \ldots, m .
\] (16)

Then \( P = \{ r_{ij}, h_{ij} \}_{i=1}^{m}, j=1, \ldots, m \) with probabilities \( p_{ij} \) defined by (15) will be a probabilistic portfolio. The main idea of the following remarks is to reduce the study of the possibilistic portfolio \( P' \) to the study of the probabilistic portfolio \( P \). Let \( f_1, \ldots, f_m \) the weights of the investor’s revenue given to assets \( A_1, \ldots, A_m \). The mean possibilistic return of portfolio \( P' \) will be defined as the mean probabilistic return of \( P \).

\[
E_P(f_1, \ldots, f_m) = \sum_{i=1}^{m} f_i E(X_i) = \sum_{i=1}^{m} f_i \bar{\mu}_i
\] (17)

The risk of asset \( A_i \) will be defined by the variance of \( X_i \):

\[
Var(X_i) = \sum_{j=1}^{n} p_{ij} (r_{ij} - \bar{\mu}_i)^2, \quad i = 1, \ldots, m .
\] (18)

The total risk of portfolio \( P' \) will be defined as the total risk of portfolio \( P \):

\[
V_{P}(f_1, \ldots, f_m) = \sum_{i,k=1}^{m} f_i f_k \text{cov}(X_i, X_k) = \sum_{i=1}^{m} f_i^2 \text{Var}(X_i) + 2 \sum_{1 \leq i < k \leq m} f_i f_k \text{cov}(X_i, X_k) .
\] (19)

By formula (19), for \( 1 \leq i < k \leq m \), the covariance \( \text{cov}(X_i, X_k) \) should be specified from the beginning. Therefore we will have as initial data the numbers \( \sigma_{ik} \) for \( 1 \leq i < k \leq m \). Consequently we will take

\[
\text{cov}(X_i, X_k) = \sigma_{ik} \quad \text{for any } 1 \leq i < k \leq m .
\] (20)
Setting an average expected level $\lambda$ of the possibilistic return of portfolio $P'$, one reaches the following optimization problem (**):

$$\min_{f_1, \ldots, f_m} V_p(f_1, \ldots, f_m)$$

$$E_p(f_1, \ldots, f_m) = \lambda$$

$$f_1 + \ldots + f_m = 1$$

$$f_1, \ldots, f_m \geq 0.$$  

We notice that (**) is exactly the optimization problem corresponding to the probabilistic portfolio $P$, but with the data offered by the possibilistic portfolio $P'$. To solve it means to find the weights $f_1, \ldots, f_m$ for which the possibilistic average expected return $\lambda$ is achieved, but with a minimum risk.

**Open problem 2** We keep the notations from the previous section. As in the paper of Huang (2009) one can formulate the following optimization problem in credibilistic terms:

$$\min_{f_1, \ldots, f_m} V[f_1 \xi_1 + \ldots + f_m \xi_m]$$

$$I \in \sum_{i=1}^{m} f_i Q(\xi_i) = \lambda$$

$$f_1 + \ldots + f_m = 1$$

$$f_1, \ldots, f_m \geq 0.$$  

The following probabilistic optimization problem can be associated with problem I:

$$\min_{f_1, \ldots, f_m} \text{Var}[f_1X_1 + \ldots + f_mX_m]$$

$$II \in \sum_{i=1}^{m} f_i E(X_i) = \lambda$$

$$f_1 + \ldots + f_m = 1$$

$$f_1, \ldots, f_m \geq 0.$$  

Problems I and II differ only in their optimum conditions. An open problem is to establish a relationship between problems I and II and then between their optimal solutions. To solve this problem one should know the relation between $\text{Var}[f_1X_1 + \ldots + f_mX_m]$ and $V[f_1\xi_1 + \ldots + f_m\xi_m]$. The optimization model from the previous section compensates the lack of an answer to the open problem formulated above.

---

**Case example: VC invests to a portfolio of start-up firms**

Suppose a venture capitalist (VC), which needs to make a selection from the set of start-up companies (assets) under analysis for further financing. A similar VC case is discussed in (Mun, 2010, pp. 603-606). The allocation of capital needs to be done to maximize the return while minimizing the risk. The problem is to construct and manage a diversified portfolio of the right firms. Traditional (probabilistic) portfolio optimization techniques work for the diversification of easy-to-value firms, but the new economy characterized by start-ups with opportunities related to novel technologies, novel business plans, and novel types of intangibles, calls for novel techniques to value the right firms. Start-up firms often have significant value even if their current cash flow situation is virtually non-existing. Their highly uncertain value largely derives from their strategic investment options related to intangibles. The intangibles can be of the traditional type, i.e., intellectual property, property rights, patents, branding, and trademarks, as well as, e-tangibles, such as marketing intangible, process and product technologies, customer loyalty, proprietary software, speed, web page hits, etc (Mun, 2010). We discuss how cash flow based valuation can be accompanied by portfolio optimization to support the decision of what proportion of the funds should be allocated to each qualified startup.
Cash-flow scenarios as a basis for possibilistic distributions

Start-up companies are not valued in the market and comparables are difficult to find. That is why we consider (fuzzy) cash-flow valuation, which requires cash-flow forecasts. Figure 1 shows an example of constructed annual cash-flow scenarios (optimistic, most likely, and pessimistic) for a start-up firm.

Fig. 1. Triangular NPV distributions of annual scenario forecasts

The construction of (3 or more) cash flow scenarios is a standard valuation method in modern corporations. The cash-flow scenario inputs could be collected as fuzzy numbers (each scenario forecast could be a fuzzy number). However, we assume here the use of single numbers for each scenario. Figure 1 shows how triangular distributions can be “fitted” to scenarios: they are interpreted as triangular fuzzy distributions, which is an approach taken, e.g., by Collan et al (2009a; 2009b) and Kinnunen (2010). The fitting could also be done differently and also simulation could be used as is done in probabilistic terms by Datar and Mathews (2004) and Mathews, Datar, and Johnson, (2007). The more there are scenarios (constructed cash-flow scenarios or simulated values) the closer we get to a continuous triangular distribution, which is seen on the left side of Figure 2. In this example we have the three discrete cash-flow scenario values as seen on the right side of Figure 2. Note that the pessimistic and optimistic values are not seen as the values with the smallest possibilities (which is the way they are presented in Figure 1, where the pessimistic value corresponds to a-\(\alpha\) and the optimistic value to a+\(\beta\)).

Fig. 2. Triangular fuzzy distribution and discrete scenario values
The total value of a target is its value without managerial flexibility plus its strategic value (Luehrman 1998, Kinnunen, 2010; Smit and Trigeorgis, 2006). Copeland and Antikarov (2001) assume that the best estimate of the market value of the project is the present value of the project without flexibility (the idea developed further by Schneider et al., 2008, and Barton and Lawryshyn, 2010). We consider the latter approach, where the total value consists of the sum of the properly discounted yearly cash flows, i.e., the cumulated forecasted future cash flows including value from tangibles, intangibles, and the terminal value, i.e., the cash flows after the forecast period. We calculate the return for venture capitalist using the total market value of a startup (assuming that the total value reflects the market value, i.e., the price at which a startup can be sold off through, e.g., a direct sale or an IPO) times the share of a startup received against VC’s investments to a startup:

\[
\left(\frac{\text{Market value of company}}{\text{VC’s share}}\right) - 1
\]

\[
\left(\frac{\text{VC’s investment expenditure}}{\text{VC’s share}}\right)
\]

A numerical example

We consider a portfolio consisting of two assets, i.e., two start-up firms, for which we have the discrete return data corresponding to the optimistic, most likely and pessimistic cash flow scenarios. The data is seen in Table 1.

Table 1. The possibilistic portfolio is made of two assets \((m=2)\) and is defined by the three discrete returns with the possibilities \((n=3)\) based on expert assessment.

<table>
<thead>
<tr>
<th>Asset (A_1)</th>
<th>Pessimistic</th>
<th>Most likely</th>
<th>Optimistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Return</td>
<td>(r_{11} = 3%)</td>
<td>(r_{12} = 6%)</td>
<td>(r_{13} = 9%)</td>
</tr>
<tr>
<td>Possibility</td>
<td>(\mu_{11} = \frac{5}{10})</td>
<td>(\mu_{12} = 1)</td>
<td>(\mu_{13} = \frac{7}{10})</td>
</tr>
<tr>
<td>Asset (A_2)</td>
<td>Pessimistic</td>
<td>Most likely</td>
<td>Optimistic</td>
</tr>
<tr>
<td>Return</td>
<td>(r_{21} = 4%)</td>
<td>(r_{22} = 5%)</td>
<td>(r_{23} = 8%)</td>
</tr>
<tr>
<td>Possibility</td>
<td>(\mu_{21} = 1)</td>
<td>(\mu_{22} = \frac{6}{10})</td>
<td>(\mu_{23} = \frac{8}{10})</td>
</tr>
</tbody>
</table>

We assume that \(\text{cov}(X_1, X_2) = 1.05\). The possibilistic returns corresponding to the two assets are the discrete possibilistic distributions:

\[
\xi_1: \begin{bmatrix} 3 & 6 & 9 \\ 5 & 1 & 7 \\ 10 & 10 & 10 \end{bmatrix}, \quad \xi_2: \begin{bmatrix} 4 & 5 & 8 \\ 1 & 6 & 8 \\ 10 & 10 & 10 \end{bmatrix}
\]

We compute the associated probabilities cf. formula (15):

\[
p_{11} = \frac{5}{20}, \quad p_{12} = \frac{8}{20}, \quad p_{13} = \frac{7}{20}, \quad p_{21} = \frac{12}{20}, \quad p_{22} = 0, \quad p_{23} = \frac{8}{20}
\]

We consider the discrete random variable \(X_i\) associated with the fuzzy variable \(\xi_i, i=1,2\):

\[
X_1: \begin{bmatrix} 3 & 6 & 9 \\ 5 & 8 & 7 \\ 20 & 20 & 20 \end{bmatrix}, \quad X_2: \begin{bmatrix} 4 & 5 & 8 \\ 12 & 0 & 8 \\ 20 & 20 & 20 \end{bmatrix}
\]

The part of the investor’s budget invested in the first asset is \(f\); and in the second asset is \(1-f\), \(f \in [0, 1]\). We turned the possibilistic portfolio problem into a probabilistic one. We compute the mean expected return for each asset:

\[
\overline{X}_1 = \frac{126}{20} \%, \quad \overline{X}_2 = \frac{112}{20} \%
\]

The variance (= the risk) for each asset:
\[ \text{Var}(X_1) = \frac{2124}{400}, \quad \text{Var}(X_2) = \frac{1536}{400} \]

The mean return of the portfolio:
\[ E(X) = \bar{X}_1 f + \bar{X}_2 (1-f) = \frac{126}{20} f + \frac{112}{20} (1-f) \]

The variance of the portfolio:
\[ \text{Var}(f) = f^2 \text{Var}(X_1) + (1 - f)^2 \text{Var}(X_2) + 2f(1-f)\text{cov}(X_1, X_2) \]

Then \[ \text{Var}(f) = \frac{2124}{400} f^2 + (1-f)^2 \frac{1536}{400} + 2f(1-f)1.05 \]

The optimization problem is:
\[
\begin{align*}
\min & \quad \text{Var}(f) \\
\text{s.t.} & \quad E(X) = \lambda \\
& \quad f \in [0,1]
\end{align*}
\]

We assume that \( \lambda = 6 \). Then from \( E(X) = 6 \) we obtain the equation in \( \lambda \):
\[ \frac{126}{20} f + \frac{112}{20} (1-f) = 6, \quad \text{and therefore} \quad f = \frac{8}{14} = 0.5714, \quad 1-f = 0.4286, \quad \text{and} \]
\[ \text{Min Var}(f) = 2.9534 \]

The results of the portfolio analysis suggests that the venture capital company should invest 57\% of its budget to firm 1, i.e., asset \( A_1 \), and 43\% to firm 2, i.e., asset \( A_2 \).

**Concluding remarks**

Risk appears in phenomena subject to uncertainty. There are several ways of representing the risk, according to the study in which the risk situation is placed or the type of uncertainty. If uncertainty is probabilistic (with events which occur a large number of times), then we talk about probabilistic risk. Alternatively, there exists the possibilistic risk, for the uncertain situations described by Zadeh’s possibility theory. The idea of this paper consists in describing some aspects of risk by credibilistic theory. The notion of credibilistic expected value and its properties allow us to pass from possibilistic models to probabilistic models in the discrete case. The paper also presented an application of a venture capitalist firm, which invests to a portfolio of start-up companies. An open problem is the approach of the contribution of the paper (possibilistic portfolio) for other classes of possibilistic distributions (larger than the discrete ones).

**Acknowledgments**— The work of Irina Georgescu was supported by CNCSIS-UEFISCSU project number PN II-RU 651/2010.

**References**


Kinnunen J (2010). Valuing M&A Synergies as (Fuzzy) Real Options. 14th Annual International Conference on Real Options, Rome, Italy, June 16-19


